Communications in Partial Differential Equations, 36: 134–161, 2011

ISSN 0360-5302 print/1532-4133 online
DOI: 10.1080/03605302.2010.498855

Regularity Results for $\bar{\partial}_b$ on CR-Manifolds of Hypersurface Type

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We introduce a class of CR-manifolds of hypersurface type called weak Y(q)-manifolds that includes Y(q) manifolds and q-pseudoconvex manifolds. We develop the L^2-regularity theory of the complex Green operator on weak Y(q) manifolds and show that $\bar{\partial}_b$ and the Kohn Laplacian have closed range at all Sobolev levels, the space of harmonic forms is finite dimensional, the Szegö kernel is continuous and $\bar{\partial}_b$ can be solved in C^∞ on the appropriate forms levels. Our argument involves building a weighted norm from a microlocal decomposition.

Keywords $\bar{\partial}_b$; Closed range; CR-manifold; Hypersurface type; Microlocal analysis; Tangential Cauchy–Riemann operator; Weak $Y(q)$; $Y(q)$.

Mathematics Subject Classification Primary 32W10; Secondary 35N10, 32V20, 35A27, 32F10.

1. Introduction and Results

In this article, we introduce a class of embedded CR manifolds satisfying a geometric condition that we call weak $Y(q)$. For such manifolds, we show that $\bar{\partial}_b$ has closed range on L^2 and that the complex Green operator is continuous on L^2. Our method involves building a weighted norm from a microlocal decomposition. We also prove that at any Sobolev level there is a weight such that the complex Green operator inverting the weighted Kohn Laplacian is continuous. Thus, we can solve the $\bar{\partial}_b$-equation in C^∞.

Let $M^{2n-1} \subset \mathbb{C}^N$ be a C^∞ compact, orientable CR-manifold, $N \geq n$. We say that M is of hypersurface type if the CR-dimension of M is $n-1$ so that the complex tangent bundle of M splits into a complex sub-bundle of dimension $n-1$, the conjugate of the complex sub-bundle, and one totally real direction. When the de Rham complex on M is restricted to the conjugate of the complex sub-bundle, we obtain the $\bar{\partial}_b$ complex.

Received December 15, 2009; Accepted April 28, 2010
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When M is the boundary of a pseudoconvex domain, closed range for $\bar{\partial}_b$ was obtained in [2, 11, 15]. This work was extended to pseudoconvex manifolds of hypersurface type by Nicoara in [12]. When the domain is not pseudoconvex, there is a condition $Y(q)$ which is known to imply subelliptic estimates for the complex Green operator acting on $(0, q)$ forms (see [6, 7] for details on $Y(q)$). In this article, we will adapt the microlocal analysis used in [12, 13] to obtain closed range results for $\bar{\partial}_b$ on manifolds satisfying weak $Y(q)$.

When M is a CR-manifold of hypersurface type, the tangent space of M can be spanned by $(1, 0)$ vector fields $L_1, \dots, L_{n-1}$, their conjugates and a purely imaginary vector field T spanning the remaining direction. If $\bar{\partial}_b^*$ denotes the Hilbert space adjoint of $\bar{\partial}_b$ with respect to the L^2 inner product on M, we have a basic identity for $(0, q)$ forms ϕ of the form

$$\|\bar{\partial}_b\phi\|^2 + \|\bar{\partial}_b^*\phi\|^2 = \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\overline{L}_j\phi_J\|^2 + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}(c_{jk}T\phi_{jI}, \phi_{kI}) + \cdots$$

where c_{jk} denotes the Levi-form of M in local coordinates (see, for example, [6, proof of Theorem 8.3.5]) and $\mathcal{I}_q$ is the set of increasing q-tuples. The difficulty in using the basic identity to prove regularity estimates for $\bar{\partial}_b$ rests in controlling the $\mathrm{Re}(c_{jk}T\phi_{jI}, \phi_{kI})$ terms. When M satisfies $Y(q)$, integration by parts can be performed on the gradient term in such a way that

$$\|\bar{\partial}_b\phi\|^2 + \|\bar{\partial}_b^*\phi\|^2 \geq C\left(\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\overline{L}_j\phi_J\|^2 + \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|L_j\phi_J\|^2\right) + \cdots.$$

Using Hörmander's classic result on sums of squares [9], this can be used to estimate $\|\phi\|_{1/2}$. On manifolds where the Levi-form degenerates, it may still be possible to choose good local coordinates so that with a suitable integration by parts, there is the estimate

$$\|\bar{\partial}_b\phi\|^2 + \|\bar{\partial}_b^*\phi\|^2 \geq \sum_{J\in\mathcal{I}_q}\sum_{j=m+1}^{n-1}\|\overline{L}_j\phi_J\|^2 + \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{m}\|L_j\phi_J\|^2 + \cdots,$$

for some integer m. Unfortunately, since such an estimate no longer bounds all of the L_j and $\overline{L}_j$ derivatives, it is not possible to control $\|\phi\|_{1/2}$. Hence, a weight function is needed to provide some positivity in the L^2-norm. The key idea in [12, 13] is to microlocalize and decompose a form ϕ into pieces whose Fourier transform is supported on specific regions. The authors then build a weighted norm based on the decomposition. In this weighted L^2-space, the $c_{jk}T$ terms are under control and a basic estimate holds. If the weight function is $t|z|^2$, then Nicoara proves that $\bar{\partial}_b$ has closed range in L^2 and in H^s, and if the weight function is obtained from property (P_q), then Raich shows that the complex Green operator is compact on $H^s(M)$ for all $s \geq 0$.

It is already known through an integration by parts argument (see the work of Ahn et al. [1] or Zampieri [17]) that local regularity estimates hold on a class of domains where the Levi-form has degeneracies and mixed signature (known as q-pseudoconvex domains). Our method is to apply microlocal analysis to the integration by parts argument used in the q-pseudoconvex case to obtain a more general sufficient condition for (global) L^2 and Sobolev space estimates.

Our main results are the following.

Theorem 1.1. *Let M^{2n-1} be a C^∞ compact, orientable weakly $Y(q)$ CR-manifold embedded in $\mathbb{C}^N$, $N \geq n$ and $1 \leq q \leq n-2$. Then the following hold*:

(i) *The operators $\bar\partial_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ and $\bar\partial_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ have closed range*;
(ii) *The operators $\bar\partial^*_b : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ and $\bar\partial^*_b : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ have closed range*;
(iii) *The Kohn Laplacian defined by $\Box_b = \bar\partial_b\bar\partial^*_b + \bar\partial^*_b\bar\partial_b$ has closed range on $L^2_{0,q}(M)$*;
(iv) *The complex Green operator G_q is continuous on $L^2_{0,q}(M)$*;
(v) *The canonical solution operators for $\bar\partial_b$, $\bar\partial^*_b G_q : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ and $G_q\bar\partial^*_b : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$, are continuous*;
(vi) *The canonical solution operators for $\bar\partial^*_b$, $\bar\partial_b G_q : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ and $G_q\bar\partial_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$, are continuous*;
(vii) *The space of harmonic forms $\mathcal{H}^q(M)$, defined to be the $(0,q)$-forms annihilated by $\bar\partial_b$ and $\bar\partial^*_b$ is finite dimensional*;
(viii) *If $\tilde q = q$ or $q+1$ and $\alpha \in L^2_{0,\tilde q}(M)$ so that $\bar\partial_b\alpha = 0$, then there exists $u \in L^2_{0,\tilde q-1}(M)$ so that*

$$\bar\partial_b u = \alpha;$$

(ix) *The Szegö projections $S_q = I - \bar\partial^*_b\bar\partial_b G_q$ and $S_{q-1} = I - \bar\partial^*_b G_q \bar\partial_b$ are continuous on $L^2_{0,q}(M)$ and $L^2_{0,q-1}(M)$, respectively.*

These results will be obtained by studying a family of weighted operators with respect to a norm $|||\phi|||_t$ defined in terms of the weights $e^{t|z|^2}$ and $e^{-t|z|^2}$ and the microlocal decomposition of ϕ. For such operators, we will also be able to obtain Sobolev space estimates, as follows:

Theorem 1.2. *Let M^{2n-1} be a C^∞ compact, orientable weakly $Y(q)$ CR-manifold embedded in $\mathbb{C}^N$, $N \geq n$. For $s \geq 0$ there exists $T_s \geq 0$ so that the following hold*:

(i) *The operators $\bar\partial_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ and $\bar\partial_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ have closed range with respect to $|||\cdot|||_t$. Additionally, for any $s > 0$ if $t \geq T_s$, then $\bar\partial_b : H^s_{0,q}(M) \to H^s_{0,q+1}(M)$ and $\bar\partial_b : H^s_{0,q-1}(M) \to H^s_{0,q}(M)$ have closed range with respect to $|||\Lambda^s\cdot|||_t$*;
(ii) *The operators $\bar\partial^*_{b,t} : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ and $\bar\partial^*_{b,t} : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ have closed range with respect to $|||\cdot|||_t$. Additionally, if $t \geq T_s$, then $\bar\partial^*_{b,t} : H^s_{0,q+1}(M) \to H^s_{0,q}(M)$ and $\bar\partial^*_{b,t} : H^s_{0,q}(M) \to H^s_{0,q-1}(M)$ have closed range with respect to $|||\Lambda^s\cdot|||_t$*;
(iii) *The Kohn Laplacian defined by $\Box_{b,t} = \bar\partial_b\bar\partial^*_{b,t} + \bar\partial^*_{b,t}\bar\partial_b$ has closed range on $L^2_{0,q}(M)$ (with respect to $|||\cdot|||_t$) and also on $H^s_{0,q}(M)$ (with respect to $|||\Lambda^s\cdot|||_t$) if $t \geq T_s$*;
(iv) *The space of harmonic forms $\mathcal{H}^q_t(M)$, defined to be the $(0,q)$-forms annihilated by $\bar\partial_b$ and $\bar\partial^*_{b,t}$ is finite dimensional*;
(v) *The complex Green operator $G_{q,t}$ is continuous on $L^2_{0,q}(M)$ (with respect to $|||\cdot|||_t$) and also on $H^s_{0,q}(M)$ (with respect to $|||\Lambda^s\cdot|||_t$) if $t \geq T_s$*;
(vi) *The canonical solution operators for $\bar\partial_b$, $\bar\partial^*_{b,t}G_{q,t} : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ and $G_{q,t}\bar\partial^*_{b,t} : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ are continuous (with respect to $|||\cdot|||_t$). Additionally, $\bar\partial^*_{b,t}G_{q,t} : H^s_{0,q}(M) \to H^s_{0,q-1}(M)$ and $G_{q,t}\bar\partial^*_{b,t} : H^s_{0,q+1}(M) \to H^s_{0,q}(M)$ are continuous (with respect to $|||\Lambda^s\cdot|||_t$) if $t \geq T_s$.*

(vii) *The canonical solution operators for* $\bar{\partial}^*_{b,t}$, $\bar{\partial}_b G_{q,t} : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ *and* $G_{q,t}\bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ *are continuous (with respect to* $|||\cdot|||_t$*). Additionally,* $\bar{\partial}_b G_{q,t} : H^s_{0,q}(M) \to H^s_{0,q+1}(M)$ *and* $G_{q,t}\bar{\partial}_b : H^s_{0,q-1}(M) \to H^s_{0,q}(M)$ *are continuous (with respect to* $|||\Lambda^s \cdot |||_t$*) if* $t \geq T_s$.

(viii) *The Szegö projections* $S_{q,t} = I - \bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}$ *and* $S_{q-1,t} = I - \bar{\partial}^*_{b,t} G_{q,t}\bar{\partial}_b$ *are continuous on* $L^2_{0,q}(M)$ *and* $L^2_{0,q-1}(M)$*, respectively and with respect to* $|||\cdot|||_t$*. Additionally, if* $t \geq T_s$*, then* $S_{q,t}$ *and* $S_{q-1,t}$ *are continuous on* $H^s_{0,q}$ *and* $H^s_{0,q-1}$ *(with respect to* $|||\Lambda^s \cdot |||_t$*), respectively.*

(ix) *If* $\tilde{q} = q$ *or* $q+1$ *and* $\alpha \in H^s_{0,q}(M)$ *so that* $\bar{\partial}_b\alpha = 0$ *and* $\alpha \perp \mathscr{H}^{\tilde{q}}_t$ *(with respect to* $|||\cdot|||_t$*), then there exists* $u \in H^s_{0,\tilde{q}-1}(M)$ *so that*

$$\bar{\partial}_b u = \alpha;$$

(x) *If* $\tilde{q} = q$ *or* $q+1$ *and* $\alpha \in C^\infty_{0,\tilde{q}}(M)$ *satisfies* $\bar{\partial}_b\alpha = 0$ *and* $\alpha \perp \mathscr{H}^{\tilde{q}}_t$ *(with respect to* $\langle\cdot,\cdot\rangle_t$*), then there exists* $u \in C^\infty_{0,\tilde{q}-1}(M)$ *so that*

$$\bar{\partial}_b u = \alpha.$$

Remark 1.3. We will see below that the proof of Theorem 1.1 follows from Theorem 1.2 and the fact that the weighted and unweighted norms are equivalent. We will see in the proof of the main theorem that the constants improve as $t \to \infty$. In particular, we will show that $|||\varphi|||^2_t \leq A_t Q_{bt}(\varphi,\varphi)$ where $A_t \to 0$ as $t \to \infty$. A (weak) consequence is that if the weight is strong enough, $\bar{\partial}$ and $\bar{\partial}^*_b$ have closed range in weighted L^2 with a constant that does not depend on the weight. In the unweighted case, this means the constants may be quite large. For a more quantitative discussion, see Remark 7.1 below.

Additionally, our results hold for any abstract CR-manifold for which a q-compatible function exists. q-compatible functions are defined in Definition 2.7. They play the analogous role here of CR-plurisubharmonic functions in [12, 13].

In Section 2, we introduce the notion of weak $Y(q)$ manifolds and q-compatible functions. In Section 3, we set up the microlocal analysis and build the weighted norm. Additionally, we compute $\bar{\partial}_b$ and $\bar{\partial}^*_b$ in local coordinates. In Section 4, we adapt the microlocal analysis in [12, 13] and prove a basic estimate: Proposition 4.1. In Section 5, we use the basic estimate to begin the study of the regularity theory for $\bar{\partial}_b$, and we prove Theorems 1.2 and 1.1 in Sections 6 and 7, respectively.

2. Definitions and Notation

2.1. *CR Manifolds and* $\bar{\partial}_b$

Definition 2.1. Let $M \subset \mathbb{C}^N$ be a C^∞ manifold of real dimension $2n-1$, $n \geq 2$. M is called a CR-manifold of hypersurface type if M is equipped with a sub-bundle $T^{1,0}(M)$ of the complexified tangent bundle $\mathbb{C}TM = TM \otimes \mathbb{C}$ so that

(i) $\dim_{\mathbb{C}} T^{1,0}(M) = n-1$;
(ii) $T^{1,0}(M) \cap T^{0,1}(M) = \{0\}$ where $T^{0,1}(M) = \overline{T^{1,0}(M)}$;
(iii) $T^{1,0}(M)$ satisfies the following integrability condition: if L_1, L_2 are smooth sections of $T^{1,0}(M)$, then so is the commutator $[L_1, L_2]$.

Since M is a submanifold of $\mathbb{C}^N$, we can generate $T_z^{1,0}(M)$ for $z \in M$ from the induced CR-structure on M as follows: set $T_z^{1,0}(M) = T_z^{1,0}(\mathbb{C}^N) \cap T_z(M) \otimes \mathbb{C}$ (under the natural inclusions). Since the complex dimension of $T_z^{1,0}(M)$ is $n-1$ for all $z \in M$, we can let $T^{1,0}(M) = \bigcup_{z\in M} T_z^{1,0}(M)$. Observe that conditions (ii) and (iii) are automatically satisfied in this case.

For the remainder of this article, M^{2n-1} is a smooth, orientable CR-manifold of hypersurface type embedded in $\mathbb{C}^N$ for some $N \geq n$. Let $\Lambda^{0,q}(M)$ be the bundle of $(0,q)$-forms on M, i.e., $\Lambda^{0,q}(M) = \bigwedge^q (T^{0,1}(M)^*)$. Denote the C^∞ sections of $\Lambda^{0,q}(M)$ by $C^\infty_{0,q}(M)$.

We construct $\bar\partial_b$ using the fact that $M \subset \mathbb{C}^N$. There is a Hermitian inner product on $\Lambda^{0,q}(M)$ given by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle_x \, dV,$$

where dV is the volume element on M and $\langle \varphi, \psi \rangle_x$ is the induced inner product on $\Lambda^{0,q}(M)$. This metric is compatible with the induced CR-structure, i.e., the vector spaces $T_z^{1,0}(M)$ and $T_z^{0,1}(M)$ are orthogonal under the inner product. The involution condition (iii) of Definition 2.1 means that $\bar\partial_b$ can be defined as the restriction of the de Rham exterior derivative d to $\Lambda^{(0,q)}(M)$. The inner product gives rise to an L^2-norm $\|\cdot\|_0$, and we also denote the closure of $\bar\partial_b$ in this norm by $\bar\partial_b$ (by an abuse of notation). In this way, $\bar\partial_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ is a well-defined, closed, densely defined operator, and we define $\bar\partial_b^* : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ to be the L^2-adjoint of $\bar\partial_b$. The Kohn Laplacian $\Box_b : L^2_{0,q}(M) \to L^2_{0,q}(M)$ is defined as

$$\Box_b = \bar\partial_b^* \bar\partial_b + \bar\partial_b \bar\partial_b^*.$$

2.2. *The Levi Form and Eigenvalue Conditions*

The induced CR-structure has a local orthonormal basis $L_1, \dots, L_{n-1}$ for the $(1,0)$-vector fields in a neighborhood U of each point $x \in M$. Let $\omega_1, \dots, \omega_{n-1}$ be the dual basis of $(1,0)$-forms that satisfy $\langle \omega_j, L_k \rangle = \delta_{jk}$. Then $\overline{L}_1, \dots, \overline{L}_{n-1}$ is a local orthonormal basis for the $(0,1)$-vector fields with dual basis $\bar\omega_1, \dots, \bar\omega_{n-1}$ in U. Also, $T(U)$ is spanned by $L_1, \dots, L_{n-1}, \overline{L}_1, \dots, \overline{L}_{n-1}$, and an additional vector field T taken to be purely imaginary (so $\overline{T} = -T$). Let γ be the purely imaginary global 1-form on M that annihilates $T^{1,0}(M) \oplus T^{0,1}(M)$ and is normalized so that $\langle \gamma, T \rangle =- 1$.

Definition 2.2. The Levi form at a point $x \in M$ is the Hermitian form given by $\langle d\gamma_x, L \wedge \overline{L'} \rangle$ where $L, L' \in T_x^{1,0}(U)$, U a neighborhood of $x \in M$.

Definition 2.3. We call M weakly pseudoconvex if there exists a form γ such that the Levi form is positive semi-definite at all $x \in M$ and strictly pseudoconvex if there is a form γ such that the Levi form is positive definite at all $x \in M$.

The following two (standard) definitions are taken from Chen and Shaw [6].

Definition 2.4. Let M be an oriented CR-manfiold of real dimension $2n-1$ with $n \geq 2$. M is said to satisfy condition $Z(q)$, $1 \leq q \leq n-1$, if the Levi form associated with M has at least $n-q$ positive eigenvalues or at least $q+1$ negative eigenvalues at every boundary point. M is said to satisfy condition $Y(q)$, $1 \leq q \leq n-1$ if the Levi form has at least either $\max\{n-q, q+1\}$ eigenvalues of the same sign or $\min\{n-q, q+1\}$ pairs of eigenvalues of opposite signs at every point on M.

Note that $Y(q)$ is equivalent to $Z(q)$ and $Z(n-1-q)$. The necessity of the symmetric requirements for $\bar{\partial}_b$ at levels q and $n-1-q$ stems from the duality between $(0, q)$-forms and $(0, n-1-q)$-forms (see [7, 14] for details).

$Z(q)$ and $Y(q)$ are classical conditions and natural extensions of strict pseudoconvexity. We wish, however, for an extension of weak pseudoconvexity. Let $P \in M$ and let U be a neighborhood of P. Then there exists an orthonormal basis $L_1, \dots, L_{n-1}$ of $T^{1,0}(U)$. By the Cartan formula (see [4, p. 14]),

$$\langle d\gamma, L_j \wedge \overline{L}_k \rangle = -\langle \gamma, [L_j, \overline{L}_k] \rangle.$$

If

$$[L_j, \overline{L}_k] = c_{jk} T \mod T^{1,0}(U) \oplus T^{0,1}(U),$$

then $\langle d\gamma, L_j \wedge \overline{L}_k \rangle = c_{jk}$. For this reason, the matrix $(c_{jk})_{1 \leq j,k \leq n-1}$ is called the Levi form with respect to $L_1, \dots, L_{n-1}$.

By weakening the definition of $Z(q)$, we obtain:

Definition 2.5. Let M be a smooth, compact, oriented CR-manifold of hypersurface type of real dimension $2n-1$. We say M satisfies $Z(q)$ weakly at P if there exists

(i) a neighborhood $U \subset M$ containing P;
(ii) an integer $m = m(U) \neq q$;
(iii) an orthonormal basis $L_1, \dots, L_{n-1}$ of $T^{1,0}(U)$ so that $\mu_1 + \cdots + \mu_q - (c_{11} + \cdots + c_{mm}) \geq 0$ on U, where $\mu_1, \dots, \mu_{n-1}$ are the eigenvalues of the Levi form in increasing order.

We say that M is weakly $Z(q)$ if M is $Z(q)$ weakly at P for all $P \in M$ and the condition $m > q$ or $m < q$ is independent of P. As above, M satisfies $Y(q)$ weakly at P if M satisfies $Z(q)$ weakly at P and $Z(n-1-q)$ weakly at P.

To see that Definition 2.5 generalizes condition $Z(q)$, choose coordinates diagonalizing c_{jk} at P so that $c_{jj}|_P = \mu_j$. If the Levi-form has at least $n-q$ positive eigenvalues, then $\mu_q > 0$, so we can let $m = q-1$ and obtain $\mu_1 + \cdots + \mu_q - (c_{11} + \cdots + c_{mm}) = \mu_q > 0$ at P. If the Levi-form has at least $q+1$ negative eigenvalues, then $\mu_{q+1} < 0$, so we can let $m = q+1$ and obtain $\mu_1 + \cdots + \mu_q - (c_{11} + \cdots + c_{mm}) = -\mu_{q+1} > 0$ at P. In either case, the sum is strictly positive at P, so the estimate extends to a neighborhood U.

The preceding argument also shows that weak-$Z(q)$ is satisfied by domains where the Levi-form has a local diagonalization with increasing entries along the diagonal and has at least $n-q$ non-negative eigenvalues or $q+1$ non-positive eigenvalues. However, diagonalizability is not necessary. Consider the hypersurface in $\mathbb{C}^5$ defined by $\rho(z) = \operatorname{Im} z_5 + |z_3|^2 + |z_4|^2 + (\operatorname{Re} z_1)(|z_1|^2 - 2|z_2|^2)$. Under the coordinates $L_j = \frac{\partial}{\partial z_j} - 2i\frac{\partial \rho}{\partial z_j}\frac{\partial}{\partial z_5}$ and $T = 2i\frac{\partial}{\partial z_5} + 2i\frac{\partial}{\partial \bar{z}_5}$ the Levi-form looks like

$$\begin{pmatrix} 2\operatorname{Re} z_1 & -z_2 & 0 & 0 \\ -\bar{z}_2 & -2\operatorname{Re} z_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We can compute the eigenvalues of this matrix in increasing order as

$$\left\{-\sqrt{4(\operatorname{Re} z_1)^2 + |z_2|^2}, \sqrt{4(\operatorname{Re} z_1)^2 + |z_2|^2}, 1, 1\right\}.$$

Since the corresponding eigenvectors are discontinuous at $P = 0$, the Levi-form cannot be diagonalized in a neighborhood of $P = 0$. In fact, we cannot even continuously separate the positive and negative eigenspaces. Let $q = 2$ and $m = 0$. The sum of the two smallest eigenvalues is zero, so this domain satisfies weak $Z(2)$, which is equivalent to weak $Y(2)$ when $n = 5$.

The signature of the Levi-form may also change locally. If we let $\rho(z) = \operatorname{Im} z_5 + |z_2|^2 + |z_3|^2 + |z_4|^2 + \operatorname{Re}((z_1)^2 \bar{z}_1)$ with L_j and T as before, then we have a diagonal Levi-form with eigenvalues $\{2\operatorname{Re}(z_1), 1, 1, 1\}$. When $\operatorname{Re}(z_1) > 0$, we have four positive eigenvalues. When $\operatorname{Re}(z_1) < 0$, we have three positive and one negative eigenvalues. Note that since we always have at least three positive eigenvalues, this satisfies the standard definition of $Y(2)$. From the standpoint of weak $Z(2)$, we can take $m = 0$ and obtain $\mu_1 + \mu_2 = 2\operatorname{Re}(z_1) + 1 > 0$ near P, or we can take $m = 1$ and obtain $\mu_1 + \mu_2 - c_{11} = (2\operatorname{Re}(z_1) + 1) - 2\operatorname{Re}(z_1) = 1 > 0$, so either value of m may work. Hence, the appropriate value of m need not be constant on M. However, since we disallow $m = q$, the condition $m < q$ or $m > q$ must be global.

If we can choose $m < q$ independent of the local neighborhood U, then weak $Z(q)$ agrees with $(q-1)$-pseudoconvexity (see [17] for the definition on boundaries of domains and further references, or [1] for generic CR submanifolds). If M satisfies weak $Z(1)$ for a choice of $m = 0$, then M is simply a weakly pseudoconvex CR-manifold of hypersurface type.

Remark 2.6. For a CR-manifold M that satisfies weak $Y(q)$, the m that corresponds to $Z(q)$ has no relation to the m that corresponds to $Z(n-1-q)$. To emphasize this, we may use m_q for the integer-valued function on M that corresponds to weak $Z(q)$ and similarly m_{n-1-q} for weak $Z(n-1-q)$.

2.3. *q-Compatible Functions*

Let $\mathcal{J}_q = \{J = (j_1, \ldots, j_q) \in \mathbb{N}^q : 1 \leq j_1 < \cdots < j_q \leq n-1\}$.

Let λ be a function defined near M and define the 2-form

$$\Theta^\lambda = \frac{1}{2}\left(\partial_b \bar{\partial}_b \lambda - \bar{\partial}_b \partial_b \lambda\right) + \frac{1}{2}\nu(\lambda) d\gamma. \tag{1}$$

where ν is the real part of the complex normal to M. We will sometimes consider Θ^λ to be the matrix $\Theta^\lambda = (\Theta^\lambda_{jk})$ where $\Theta^\lambda_{jk} = \langle \Theta^\lambda, L_j \wedge \overline{L_k} \rangle$.

Definition 2.7. Let M be a smooth, compact, oriented CR-manifold of hypersurface type of real dimension $2n-1$ satisfying $Z(q)$ weakly at some point $P \in M$. Let λ be a smooth function near M. We say λ is q-compatible with M at P if there exists a neighborhood $U \subset M$ containing P, an integer $m = m_q(U)$ from weak $Z(q)$, an orthonormal basis $L_1, \dots, L_{n-1}$ of $T^{1,0}(U)$, and a constant $B_\lambda > 0$ satisfying

(i) $\mu_1 + \cdots + \mu_q - (c_{11} + \cdots + c_{mm}) \geq 0$ on U, where $\mu_1, \dots, \mu_{n-1}$ are the eigenvalues of the Levi form in increasing order.
(ii) $b_1 + \cdots + b_q - (\Theta_{11} + \cdots + \Theta_{mm}) \geq B_\lambda$ on U if $m < q$, where $b_1, \dots, b_{n-1}$ are the eigenvalues of Θ in increasing order.
(iii) $b_{n-q} + \cdots + b_{n-1} - (\Theta_{11} + \cdots + \Theta_{mm}) \leq -B_\lambda$ on U if $m > q$.

We call B_λ the positivity constant of λ. Observe that if M is pseudoconvex, M satisfies Definition 2.5 for any $1 \leq q \leq n-1$ and any orthonormal basis $L_1, \dots, L_{n-1}$ by selecting $m = 0$. Hence, plurisubharmonic functions will be q-compatible with pseudoconvex domains for any $1 \leq q \leq n-1$.

Remark 2.8. If $\lambda = |z|^2$ then Proposition 3.1 below proves that $\Theta = \partial\bar{\partial}$ when tested against complex tangent vectors of M. Tested against such vectors, $\Theta^{|z|^2} = I$. Since this is diagonal and all of the eigenvalues of I are 1, $b_1 + \cdots + b_q - (\Theta_{11} + \cdots + \Theta_{mm}) = q - m \geq 1$ if $q > m$ and $b_{n-q} + \cdots + b_{n-1} - (\Theta_{11} + \cdots + \Theta_{mm}) = q - m \leq -1$ if $q < m$. Hence, $\lambda = |z|^2$ is always a q-compatible function on M with positivity constant 1.

Remark 2.9. Without the requirement that $\{L_1, \dots, L_{n-1}\}$ are orthonormal, $\lambda = |z|^2$ may not be a q-compatible function for all values of $m \neq q$. For a given choice of non-orthonormal local coordinates, we can always define a local function which is q-compatible for all allowable q and m, but there is no guarantee that such local functions could be made global. Hence, if we remove the restriction that the local coordinates in Definition 2.7 are orthonormal, we must also assume the existence of a global function which is q-compatible for all allowable choices of q and m.

Remark 2.10. We note that if for every $B_\lambda > 0$ there exists a q-compatible function λ satisfying $0 \leq \lambda \leq 1$ with positivity constant B_λ, then the methods of [13] can be incorporated into our current paper to show that the complex Green operator is compact. Such a condition is analogous to Catlin's Property (P) [5].

In this article, constants with no subscripts may depend on n, N, M but not any relevant q-compatible function. Those constants will be denoted with an appropriate subscript. The constant A will be reserved for the constant in the construction of the pseudodifferential operator in Section 3.

3. Computations in Local Coordinates

3.1. *Local Coordinates and CR-Plurisubharmonicity*

The following result is proved in [13].

Proposition 3.1. *Let M^{2n-1} be a smooth, orientable CR-manifold of hypersurface type embedded in $\mathbb{C}^N$ for some $N \geq n$. If λ is a smooth function near M, $L \in T^{1,0}(M)$, and v*

is the real part of the complex normal to M, then on M

$$\left\langle \frac{1}{2}\left(\partial\bar{\partial}\lambda - \bar{\partial}\partial\lambda\right), L \wedge \overline{L} \right\rangle - \left\langle \frac{1}{2}\left(\partial_b\bar{\partial}_b\lambda - \bar{\partial}_b\partial_b\lambda\right), L \wedge \overline{L} \right\rangle = \frac{1}{2}\nu\{\lambda\}\langle d\gamma, L \wedge \overline{L}\rangle$$

3.2. *Pseudodifferential Operators*

We follow the setup for the microlocal analysis in [13]. Since M is compact, there exists a finite cover $\{U_\nu\}_\nu$ so each U_ν has a special boundary system and can be parameterized by a hypersurface in $\mathbb{C}^n$ (U_ν may be shrunk as necessary). To set up the microlocal analysis, we need to define the appropriate pseudodifferential operators on each U_ν. Let $\xi = (\xi_1, \dots, \xi_{2n-2}, \xi_{2n-1}) = (\xi', \xi_{2n-1})$ be the coordinates in Fourier space so that ξ' is dual to the part of $T(M)$ in the maximal complex subspace $(T^{1,0}(M) \oplus T^{0,1}(M))$ and ξ_{2n-1} is dual to the totally real part of $T(M)$, i.e., the "bad" direction T. Define

$$\begin{aligned}
\mathscr{C}^+ &= \left\{\xi : \xi_{2n-1} \geq \frac{1}{2}|\xi'| \text{ and } |\xi| \geq 1\right\};\\
\mathscr{C}^- &= \{\xi : -\xi \in \mathscr{C}^+\};\\
\mathscr{C}^0 &= \left\{\xi : -\frac{3}{4}|\xi'| \leq \xi_{2n-1} \leq \frac{3}{4}|\xi'|\right\} \cup \{\xi : |\xi| \leq 1\}.
\end{aligned}$$

Note that $\mathscr{C}^+$ and $\mathscr{C}^-$ are disjoint, but both intersect $\mathscr{C}^0$ nontrivially. Next, we define smooth functions on $\{|\xi| : |\xi|^2 = 1\}$. Let

$$\begin{aligned}
&\psi^+(\xi) = 1 \quad \text{when } \xi_{2n-1} \geq \frac{3}{4}|\xi'| \quad \text{and} \quad \operatorname{supp}\psi^+ \subset \left\{\xi : \xi_{2n-1} \geq \frac{1}{2}|\xi'|\right\};\\
&\psi^-(\xi) = \psi^+(-\xi);\\
&\psi^0(\xi) \quad \text{satisfies } \psi^0(\xi)^2 = 1 - \psi^+(\xi)^2 - \psi^-(\xi)^2.
\end{aligned}$$

Extend ψ^+, ψ^-, and ψ^0 homogeneously outside of the unit ball, i.e., if $|\xi| \geq 1$, then

$$\psi^+(\xi) = \psi^+(\xi/|\xi|), \;\; \psi^-(\xi) = \psi^-(\xi/|\xi|), \quad \text{and} \quad \psi^0(\xi) = \psi^0(\xi/|\xi|).$$

Also, extend ψ^+, ψ^-, and ψ^0 smoothly inside the unit ball so that $(\psi^+)^2 + (\psi^-)^2 + (\psi^0)^2 = 1$. Finally, for a fixed constant $A > 0$ to be chosen later, define for any $t > 0$

$$\psi_t^+(\xi) = \psi^+(\xi/(tA)), \;\; \psi_t^-(\xi) = \psi^-(\xi/(tA)), \quad \text{and} \quad \psi_t^0(\xi) = \psi^0(\xi/(tA)).$$

Next, let Ψ_t^+, Ψ_t^-, and Ψ^0 be the pseudodifferential operators of order zero with symbols ψ_t^+, ψ_t^-, and ψ_t^0, respectively. The equality $(\psi_t^+)^2 + (\psi_t^-)^2 + (\psi_t^0)^2 = 1$ implies that

$$(\Psi_t^+)^*\Psi_t^+ + (\Psi_t^0)^*\Psi_t^0 + (\Psi_t^-)^*\Psi_t^- = Id.$$

We will also have use for pseudodifferential operators that "dominate" a given pseudodifferential operator. Let ψ be cut-off function and $\tilde{\psi}$ be another cut-off function so that $\tilde{\psi}|_{\text{supp}\psi} \equiv 1$. If Ψ and $\widetilde{\Psi}$ are pseudodifferential operators with symbols ψ and $\tilde{\psi}$, respectively, then we say that $\widetilde{\Psi}$ dominates Ψ.

For each U_ν, we can define Ψ_t^+, Ψ_t^-, and Ψ_t^0 to act on functions or forms supported in U_ν, so let $\Psi_{\nu,t}^+$, $\Psi_{\nu,t}^-$, and $\Psi_{\nu,t}^0$ $\Psi_{\nu,t}^0$ be the pseudodifferential operators of order zero defined on U_ν, and let $\mathscr{C}_\nu^+$, $\mathscr{C}_\nu^-$, and $\mathscr{C}_\nu^0$ be the regions of ξ-space dual to U_ν on which the symbol of each of those pseudodifferential operators is supported. Then it follows that:

$$(\Psi_{\nu,t}^+)^*\Psi_{\nu,t}^+ + (\Psi_{\nu,t}^0)^*\Psi_{\nu,t}^0 + (\Psi_{\nu,t}^-)^*\Psi_{\nu,t}^- = Id.$$

Let $\widetilde{\Psi}_{\mu,t}^+$ and $\widetilde{\Psi}_{\mu,t}^-$ be pseudodifferential operators that dominate $\Psi_{\mu,t}^+$ and $\Psi_{\mu,t}^-$, respectively (where $\Psi_{\mu,t}^+$ and $\Psi_{\mu,t}^-$ are defined on some U_μ). If $\widetilde{C}_\mu^+$ and $\widetilde{C}_\mu^-$ are the supports of $\widetilde{\Psi}_{\mu,t}^+$ and $\widetilde{\Psi}_{\mu,t}^-$, respectively, then we can choose $\{U_\mu\}$, $\tilde{\psi}_{\mu,t}^+$, and $\tilde{\psi}_{\mu,t}^-$ so that the following result holds.

Lemma 3.2. *Let M be a compact, orientable, embedded CR-manifold. There is a finite open covering $\{U_\mu\}_\mu$ of M so that if $U_\mu, U_\nu \in \{U_\mu\}$ have nonempty intersection, then there exists a diffeomorphism ϑ between U_ν and U_μ with Jacobian $\mathscr{J}_\vartheta$ so that*:

(i) ${}^t\mathscr{J}_\vartheta(\widetilde{\mathscr{C}}_\mu^+) \cap \mathscr{C}_\nu^- = \emptyset$ *and* $\mathscr{C}_\nu^+ \cap {}^t\mathscr{J}_\vartheta(\widetilde{\mathscr{C}}_\mu^-) = \emptyset$ *where* ${}^t\mathscr{J}_\vartheta$ *is the inverse of the transpose of* $\mathscr{J}_\vartheta$;

(ii) *Let* ${}^\vartheta\Psi_{\mu,t}^+$, ${}^\vartheta\Psi_{\mu,t}^-$, *and* ${}^\vartheta\Psi_{\mu,t}^0$ *be the transfers of* $\Psi_{\mu,t}^+$, $\Psi_{\mu,t}^-$, *and* $\Psi_{\mu,t}^0$, *respectively via* ϑ. *Then on* $\{\xi : \xi_{2n-1} \geq \frac{4}{5}|\xi'| \text{ and } |\xi| \geq (1+\epsilon)tA\}$, *the principal symbol of* ${}^\vartheta\Psi_{\mu,t}^+$ *is identically* 1, *on* $\{\xi : \xi_{2n-1} \leq -\frac{4}{5}|\xi'| \text{ and } |\xi| \geq (1+\epsilon)tA\}$, *the principal symbol of* ${}^\vartheta\Psi_{\mu,t}^-$ *is identically* 1, *and on* $\{\xi : -\frac{1}{3}\xi_{2n-1} \geq \frac{1}{3}|\xi'| \text{ and } |\xi| \geq (1+\epsilon)tA\}$, *the principal symbol of* ${}^\vartheta\Psi_{\mu,t}^0$ *is identically* 1, *where* $\epsilon > 0$ *can be very small*;

(iii) *Let* ${}^\vartheta\widetilde{\Psi}_{\mu,t}^+$, ${}^\vartheta\widetilde{\Psi}_{\mu,t}^-$ *be the transfers via* ϑ *of* $\widetilde{\Psi}_{\mu,t}^+$ *and* $\widetilde{\Psi}_{\mu,t}^-$, *respectively. Then the principal symbol of* ${}^\vartheta\widetilde{\Psi}_{\mu,t}^+$ *is identically* 1 *on* $\mathscr{C}_\nu^+$ *and the principal symbol of* ${}^\vartheta\widetilde{\Psi}_{\mu,t}^-$ *is identically* 1 *on* $\mathscr{C}_\nu^-$;

(iv) $\widetilde{\mathscr{C}}_\nu^+ \cap \widetilde{\mathscr{C}}_\nu^- = \emptyset$.

We will suppress the left superscript ϑ as it should be clear from the context which pseudodifferential operator must be transferred. The proof of this lemma is contained in Lemma 4.3 and its subsequent discussion in [12].

If P is any of the operators $\Psi_{\mu,t}^+$, $\Psi_{\mu,t}^-$, or $\Psi_{\mu,t}^0$, then it is immediate that

$$D_\xi^\alpha \sigma(P) = \frac{1}{|t|^\alpha} q_\alpha(x, \xi) \tag{2}$$

for $|\alpha| \geq 0$, where $q_\alpha(x, \xi)$ is bounded independently of t.

3.3. *Norms*

We have a volume form dV on M, and we define the following inner products and norms on functions (with their natural generalizations to forms). Let λ be a smooth function defined near M. We define

$$(\phi, \varphi)_\lambda = \int_M \phi\bar{\varphi}\, e^{-\lambda} dV, \quad \text{and} \quad \|\varphi\|_\lambda^2 = (\varphi, \varphi)_\lambda$$

In particular, $(\phi, \varphi)_0 = \int_M \phi\bar{\varphi}\, dV$ and $\|\varphi\|_0^2 = (\varphi, \varphi)_0$ are the standard (unweighted) L^2 inner product and norm. If $\varphi = \sum_{J\in\mathcal{I}_q} \varphi_J \bar{\omega}_J$, then we use the common shorthand $\|\varphi\| = \sum_{J\in\mathcal{I}_q} \|\varphi_J\|$ where $\|\cdot\|$ represents any norm of φ.

We also need a norm that is well-suited for the microlocal arguments. Let λ^+ and λ^- be smooth functions defined near M. Let $\{\zeta_\nu\}$ be a partition of unity subordinate to the covering $\{U_\nu\}$ satisfying $\sum_\nu \zeta_\nu^2 = 1$. Also, for each ν, let $\tilde{\zeta}_\nu$ be a cutoff function that dominates ζ_ν so that $\operatorname{supp}\tilde{\zeta}_\nu \subset U_\nu$. Then we define the global inner product and norm as follows:

$$\begin{aligned}\langle \phi, \varphi\rangle_{\lambda^+,\lambda^-} &= \langle \phi, \varphi\rangle_\pm \\ &= \sum_\nu \Big[(\tilde{\zeta}_\nu \Psi^+_{\nu,t}\zeta_\nu\phi^\nu, \tilde{\zeta}_\nu \Psi^+_{\nu,t}\zeta_\nu\varphi^\nu)_{\lambda^+} + (\tilde{\zeta}_\nu \Psi^0_{\nu,t}\zeta_\nu\phi^\nu, \tilde{\zeta}_\nu \Psi^0_{\nu,t}\zeta_\nu\varphi^\nu)_0 \\ &\quad + (\tilde{\zeta}_\nu \Psi^-_{\nu,t}\zeta_\nu\phi^\nu, \tilde{\zeta}_\nu \Psi^-_{\nu,t}\zeta_\nu\varphi^\nu)_{\lambda^-} \Big]\end{aligned}$$

and

$$|||\varphi|||^2_{\lambda^+,\lambda} = |||\varphi|||^2_\pm = \sum_\nu \Big[\|\tilde{\zeta}_\nu \Psi^+_{\nu,t}\zeta_\nu\varphi^\nu\|^2_{\lambda^+} + \|\tilde{\zeta}_\nu \Psi^0_{\nu,t}\zeta_\nu\varphi^\nu\|^2_0 + \|\tilde{\zeta}_\nu \Psi^-_{\nu,t}\zeta_\nu\varphi^\nu\|^2_{\lambda^-} \Big],$$

where φ^ν is the form φ expressed in the local coordinates on U_ν. The superscript ν will often be omitted.

For a form φ supported on M, the Sobolev norm of order s is given by the following:

$$\|\varphi\|_s^2 = \sum_\nu \|\tilde{\zeta}_\nu \Lambda^s \zeta_\nu \varphi^\nu\|_0^2$$

where Λ is defined to be the pseudodifferential operator with symbol $(1+|\xi|^2)^{1/2}$.

In [13], it is shown that there exist constants $c_\pm$ and $C_\pm$ so that

$$c_\pm \|\varphi\|_0^2 \le |||\varphi|||^2_{\lambda^+,\lambda} \le C_\pm \|\varphi\|_0^2 \tag{3}$$

where $c_\pm$ and $C_\pm$ depend on $\max_M\{|\lambda^+| + |\lambda^-|\}$ (assuming $tA \ge 1$). Additionally, there exists an invertible self-adjoint operator $H_\pm$ so that $(\phi, \varphi)_0 = \langle \phi, H_\pm \varphi\rangle_\pm$.

3.4. $\bar{\partial}_b$ *and Its Adjoints*

If g is a function on M, in local coordinates,

$$\bar{\partial}_b g = \sum_{j=1}^{n-1} \bar{L}_j g \bar{\omega}_j,$$

while if φ is a $(0, q)$-form, there exist functions m_K^J so that

$$\bar{\partial}_b \varphi = \sum_{\substack{J\in\mathcal{I}_q \\ K\in\mathcal{I}_{q+1}}} \sum_{j=1}^{n-1} \epsilon_K^{jJ} \bar{L}_j \varphi_J \bar{\omega}_K + \sum_{\substack{J\in\mathcal{I}_q \\ K\in\mathcal{I}_{q+1}}} \varphi_J m_K^J \bar{\omega}_K$$

where ϵ_K^{jJ} is 0 if $\{j\} \cup J \neq K$ as sets and is the sign of the permutation that reorders jJ as K. We also define

$$\varphi_{jI} = \sum_{J \in \mathcal{J}_q} \epsilon_J^{jI} \varphi_J$$

(in this case, $|I| = q - 1$ and $|J| = q$). Let $\overline{L}_j^*$ be the adjoint of $\overline{L}_j$ in $(\cdot, \cdot)_0$, $\overline{L}_j^{*,\lambda}$ be the adjoint of $\overline{L}_j$ in $(\cdot, \cdot)_\lambda$. We define $\bar{\partial}_b^*$ and $\bar{\partial}_b^{*,\lambda}$ in $L^2(M)$ and $L^2(M, e^{-\lambda})$, respectively. In this paper, λ stands for λ^+ or λ^- and we will abbreviate $\bar{\partial}_b^{*,\lambda^+}$ by $\bar{\partial}_b^{*,+}$ and similarly for $\bar{\partial}_b^{*,-}$, $\overline{L}^{*,+}$, $\overline{L}^{*,-}$, etc.

On a $(0, q)$-form φ, we have (for some functions $f_j \in C^\infty(U)$ independent of φ)

$$\begin{aligned} \bar{\partial}_b^* \varphi &= \sum_{I \in \mathcal{J}_{q-1}} \sum_{j=1}^{n-1} \overline{L}_j^* \varphi_{jI} \bar{\omega}_I + \sum_{\substack{I \in \mathcal{J}_{q-1} \\ J \in \mathcal{J}_q}} \overline{m_J^I} \varphi_J \bar{\omega}_I \\ &= -\sum_{I \in \mathcal{J}_{q-1}} \sum_{j=1}^{n-1} \big(L_j \varphi_{jI} + f_j \varphi_{jI}\big) \bar{\omega}_I + \sum_{\substack{I \in \mathcal{J}_{q-1} \\ J \in \mathcal{J}_q}} \overline{m_J^I} \varphi_J \bar{\omega}_I \\ \bar{\partial}_b^{*,\lambda} \varphi &= \sum_{I \in \mathcal{J}_{q-1}} \sum_{j=1}^{n-1} \overline{L}_j^{*,\lambda} \varphi_{jI} \bar{\omega}_I + \sum_{I \in \mathcal{J}_{q-1}} \overline{m_J^I} \varphi_J \bar{\omega}_I \\ &= -\sum_{I \in \mathcal{J}_{q-1}} \sum_{j=1}^{n-1} \big(L_j \varphi_{jI} - L_j \lambda \varphi_{jI} + f_j \varphi_{jI}\big) \bar{\omega}_I + \sum_{\substack{I \in \mathcal{J}_{q-1} \\ J \in \mathcal{J}_q}} \overline{m_J^I} \varphi_J \bar{\omega}_I. \end{aligned} \tag{4}$$

Consequently, we see that

$$\bar{\partial}_b^{*,\lambda} = \bar{\partial}_b^* - [\bar{\partial}_b^*, \lambda],$$

and both adjoints have the same domain. Finally, let $\bar{\partial}_{b,\pm}^*$ be the adjoint of $\bar{\partial}_b$ with respect to $\langle \cdot, \cdot \rangle_\pm$.

The computations proving Lemmas 4.8 and 4.9 and equation (4.4) in [12] can be applied here with only a change of notation, so we have the following two results, recorded here as Lemmas 3.3 and 3.4. The meaning of the results is that $\bar{\partial}_{b,\pm}^*$ acts like $\bar{\partial}_b^{*,+}$ for forms whose support is basically $\mathscr{C}^+$ and $\bar{\partial}_b^{*,-}$ on forms whose support is basically $\mathscr{C}^-$.

Lemma 3.3. *On smooth* $(0, q)$*-forms,*

$$\begin{aligned} \bar{\partial}_{b,\pm}^* &= \bar{\partial}_b^* - \sum_\mu \zeta_\mu^2 \widetilde{\Psi}_{\mu,t}^+ [\bar{\partial}_b^*, \lambda^+] + \sum_\mu \zeta_\mu^2 \widetilde{\Psi}_{\mu,t}^- [\bar{\partial}_b^*, \lambda^-] \\ &\quad + \sum_\mu \Big(\tilde{\zeta}_\mu [\tilde{\zeta}_\mu \Psi_{\mu,t}^+ \zeta_\mu, \bar{\partial}_b]^* \tilde{\zeta}_\mu \Psi_{\mu,t}^+ \zeta_\mu + \zeta_\mu (\Psi_{\mu,t}^+)^* \tilde{\zeta}_\mu [\bar{\partial}_b^{*,+}, \tilde{\zeta}_\mu \Psi_{\mu,t}^+ \zeta_\mu] \tilde{\zeta}_\mu \\ &\quad + \tilde{\zeta}_\mu [\tilde{\zeta}_\mu \Psi_{\mu,t}^- \zeta_\mu, \bar{\partial}_b]^* \tilde{\zeta}_\mu \Psi_{\mu,t}^- \zeta_\mu + \zeta_\mu (\Psi_{\mu,t}^+)^* \tilde{\zeta}_\mu [\bar{\partial}_b^{*,-}, \tilde{\zeta}_\mu \Psi_{\mu,t}^- \zeta_\mu] \tilde{\zeta}_\mu + E_A \Big), \end{aligned}$$

where the error term E_A *is a sum of order zero terms and "lower order" terms. Also, the symbol of* E_A *is supported in* $\mathscr{C}_\mu^0$ *for each* μ*.*

We are now ready to define the energy forms that we use. Let

$$Q_{b,\pm}(\phi,\varphi) = \langle \bar\partial_b\phi, \bar\partial_b\varphi\rangle_\pm + \langle \bar\partial^*_{b,\pm}\phi, \bar\partial^*_{b,\pm}\varphi\rangle_\pm$$
$$Q_{b,+}(\phi,\varphi) = (\bar\partial_b\phi, \bar\partial_b\varphi)_{\lambda^+} + (\bar\partial^{*,+}_b\phi, \bar\partial^{*,+}_b\varphi)_{\lambda^+}$$
$$Q_{b,0}(\phi,\varphi) = (\bar\partial_b\phi, \bar\partial_b\varphi)_0 + (\bar\partial^*_b\phi, \bar\partial^*_b\varphi)_0$$
$$Q_{b,-}(\phi,\varphi) = (\bar\partial_b\phi, \bar\partial_b\varphi)_{\lambda^-} + (\bar\partial^{*,-}_b\phi, \bar\partial^{*,-}_b\varphi)_{\lambda^-}.$$

Lemma 3.4. *If φ is a smooth $(0,q)$-form on M, then there exist constants K, $K_\pm$ and K' with $K\geq 1$ so that*

$$\begin{aligned} KQ_{b,\pm}(\varphi,\varphi) &+ K_\pm\sum_\nu \|\tilde\zeta_\nu\widetilde\Psi^0_{\nu,t}\zeta_\nu\varphi^\nu\|_0^2 + K'\|\varphi\|_0^2 + O_t(\|\varphi\|^2_{-1}) \\ &\geq \sum_\nu \Big[Q_{b,+}(\tilde\zeta_\nu\Psi^+_{\nu,t}\zeta_\nu\varphi^\nu, \tilde\zeta_\nu\Psi^+_{\nu,t}\zeta_\nu\varphi^\nu) \\ &\quad + Q_{b,0}(\tilde\zeta_\nu\Psi^0_{\nu,t}\zeta_\nu\varphi^\nu, \tilde\zeta_\nu\Psi^0_{\nu,t}\zeta_\nu\varphi^\nu) + Q_{b,-}(\tilde\zeta_\nu\Psi^-_{\nu,t}\zeta_\nu\varphi^\nu, \tilde\zeta_\nu\Psi^-_{\nu,t}\zeta_\nu\varphi^\nu)\Big] \end{aligned} \quad (5)$$

K and K' do not depend on t, λ^- or λ^+.

Also, since $\bar\partial^{*,\lambda}_b = \bar\partial^*_b +$ "lower order" and $\Psi^\lambda_{\mu,t}$ satisfies (2), commuting $\bar\partial^{*,\lambda}_b$ by $\Psi^\lambda_{\mu,t}$ creates error terms of order 0 that do not depend on t or λ, although the lower order terms may themselves depend on t and λ.

4. The Basic Estimate

The goal of this section is to prove a basic estimate for smooth forms on M.

Proposition 4.1. *Let $M\subset\mathbb{C}^N$ be a compact, orientable CR-manifold of hypersurface type of dimension $2n-1$ and $1\leq q\leq n-2$. Assume that M admits functions λ_1 and λ_2 where λ_1 is a q-compatible function and λ_2 is an $(n-1-q)$-compatible function with positivity constants B_{λ^+} and B_{λ^-}, respectively. Let $\varphi\in\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial^*_b)$. Set*

$$\lambda^+ = \begin{cases} t\lambda_1 & \text{if } m_q < q \\ -t\lambda_1 & \text{if } m_q > q \end{cases}$$

and

$$\lambda^- = \begin{cases} -t\lambda_2 & \text{if } m_{n-1-q} < n-1-q \\ t\lambda_2 & \text{if } m_{n-1-q} > n-1-q \end{cases}.$$

There exist constants K, $K_\pm$, and $K'_\pm$ where K does not depend on λ^+ and λ^- so that

$$tB_\pm|||\varphi|||^2_\pm \leq KQ_{b,\pm}(\varphi,\varphi) + K|||\varphi|||^2_\pm + K_\pm\sum_\nu\sum_{J\in\mathcal{I}_q}\|\tilde\zeta_\nu\widetilde\Psi^0_{\nu,t}\zeta_\nu\varphi^\nu_J\|_0^2 + K'_\pm\|\varphi\|^2_{-1}.$$

The constant $B_\pm = \min\{B_{\lambda^+}, B_{\lambda^-}\}$.

For Theorem 1.1, we will use $\lambda_1=\lambda_2=|z|^2$.

4.1. *Local Estimates*

The crucial multilinear algebra that we need is contained in the following lemma from Straube [16]:

Lemma 4.2. *Let $B = (b_{jk})_{1\le j,k\le n}$ be a Hermitian matrix and $1 \le q \le n$. The following are equivalent*:

(i) *If $u \in \Lambda^{(0,q)}$, then $\sum_{K\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n} b_{jk}u_{jK}\overline{u_{kK}} \ge M|u|^2$.*
(ii) *The sum of any q eigenvalues of B is at least M.*
(iii) *$\sum_{s=1}^{q}\sum_{j,k=1}^{n} b_{jk}t_j^s\overline{t_k^s} \ge M$ whenever $t^1,\dots,t^q$ are orthonormal in $\mathbb{C}^n$.*

We work on a fixed $U = U_\nu$. On this neighborhood, as above, there exists an orthonormal basis of vector fields $L_1,\dots,L_n,\overline{L}_1,\dots,\overline{L}_n$ so that

$$[L_j,\overline{L}_k] = c_{jk}T + \sum_{\ell=1}^{n-1}(d_{jk}^{\ell}L_\ell - \bar{d}_{kj}^{\ell}\overline{L}_\ell) \tag{6}$$

if $1 \le j,k \le n-1$, and $T = L_n - \overline{L}_n$. Note that c_{jk} are the coefficients of the Levi form. Recall that $\overline{L}^{*,+}$, $\overline{L}^{*}$, and $\overline{L}^{*,-}$ are the adjoints of $\overline{L}$ in $(\cdot,\cdot)_{\lambda^+}$, $(\cdot,\cdot)_0$, and $(\cdot,\cdot)_{\lambda^-}$, respectively. From (4), we see that

$$\overline{L}_j^{*,\lambda} = -L_j + L_j\lambda - f_j$$

and plugging this into (6), we have

$$[\overline{L}_j^{*,\lambda},\overline{L}_k] = -c_{jk}T + \sum_{\ell=1}^{n-1}\left(d_{jk}^{\ell}(\overline{L}_\ell^{*,\lambda} - L_\ell\lambda + f_\ell) + \bar{d}_{kj}^{\ell}\overline{L}_\ell\right) - \overline{L}_kL_j\lambda + \overline{L}_kf_j. \tag{7}$$

Because of Lemma 3.4, we may turn our attention to the quadratic

$$Q_{b,\lambda}(\varphi,\varphi) = (\bar\partial_b\varphi,\bar\partial_b\varphi)_\lambda + (\bar\partial_b^{*,\lambda}\varphi,\bar\partial_b^{*,\lambda}\varphi)_\lambda.$$

We introduce the error term

$$E(\varphi) \le C\left(\|\varphi\|_\lambda^2 + \sum_{j=1}^{n-1}|(h\overline{L}_j\varphi,\varphi)_\lambda|\right) = C\left(\|\varphi\|_\lambda^2 + \sum_{j=1}^{n-1}|(\tilde{h}\overline{L}_j^{*,\lambda}\varphi,\varphi)_\lambda|\right)$$

where the operators $\overline{L}_j$ and $\overline{L}_j^{*,\lambda}$ act componentwise, C is a constant independent of φ and λ, and h and $\tilde{h}$ are bounded functions that are independent of t, A, λ^+, λ^-, and the other quantities that are carefully minding. Recall the definition that $\varphi_{jK} = \sum_{J\in\mathcal{I}_q}\epsilon_J^{jK}\varphi_J$. As in the proof of Lemma 4.2 in [13], we compute that for smooth φ supported in a sufficiently small neighborhood,

$$Q_{b,\lambda}(\varphi,\varphi) = \sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\|\overline{L}_j\varphi_J\|_\lambda^2 + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\big(c_{jk}T\varphi_{jI},\varphi_{kI}\big)_\lambda + E(\varphi)$$

$$+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left\{\frac{1}{2}\big((\overline{L}_jL_k\lambda+L_j\overline{L}_k\lambda)\varphi_{jI},\varphi_{kI}\big)_\lambda\right.$$
$$\left.+\frac{1}{2}\sum_{\ell=1}^{n-1}\big((d^\ell_{jk}L_\ell\lambda+\overline{d^\ell_{jk}}\overline{L}_\ell\lambda)\varphi_{jI},\varphi_{kI}\big)_\lambda\right\}. \tag{8}$$

The weak $Z(q)$-hypothesis suggests that we ought to integrate by parts to take advantage of the positivity/negativity conditions. By (7) and integration by parts, we have

$$\|\overline{L}_j\varphi_J\|^2_\lambda-\|\overline{L}^{*,\lambda}_j\varphi_J\|^2_\lambda=-\mathrm{Re}(c_{jj}T\varphi_J,\varphi_J)-\sum_{\ell=1}^{n-1}\mathrm{Re}\big(d^\ell_{jj}(L_\ell\lambda)\varphi_J,\varphi_J\big)$$
$$-\mathrm{Re}((\overline{L}_jL_j\lambda)\varphi_J,\varphi_J)+E(\varphi). \tag{9}$$

Consequently, we can use (7) and (9) to obtain

$$Q_{b,\lambda}(\varphi,\varphi)$$
$$=\sum_{J\in\mathcal{I}_q}\left\{\sum_{j=1}^{m}\|\overline{L}^{*,\lambda}_j\varphi_J\|^2_\lambda+\sum_{j=m+1}^{n-1}\|\overline{L}_j\varphi_J\|^2_\lambda\right\}+E(\varphi)$$
$$+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\big(c_{jk}T\varphi_{jI},\varphi_{kI}\big)_\lambda-\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{m}\mathrm{Re}\big(c_{jj}T\varphi_J,\varphi_J\big)_\lambda$$
$$+\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left\{\frac{1}{2}\big((\overline{L}_jL_k\lambda+L_j\overline{L}_k\lambda)\varphi_{jI},\varphi_{kI}\big)_\lambda+\frac{1}{2}\sum_{\ell=1}^{n-1}\big((d^\ell_{jk}L_\ell\lambda+\overline{d^\ell_{jk}}\overline{L}_\ell\lambda)\varphi_{jI},\varphi_{kI}\big)_\lambda\right\}$$
$$-\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{m}\left\{\frac{1}{2}\big((\overline{L}_jL_j\lambda+L_j\overline{L}_j\lambda)\varphi_J,\varphi_J\big)_\lambda+\frac{1}{2}\sum_{\ell=1}^{n-1}\big((d^\ell_{jj}L_\ell\lambda+\overline{d^\ell_{jj}}\overline{L}_\ell\lambda)\varphi_J,\varphi_J\big)_\lambda\right\}. \tag{10}$$

We are now in a position to control the "bad" direction terms. Recall the following consequence of the sharp Gårding inequality from [13].

Proposition 4.3. *Let R be a first order pseudodifferential operator such that $\sigma(R)\geq\kappa$ where κ is some positive constant and (h_{jk}) a hermitian matrix (that does not depend on ξ). Then there exists a constant C such that if the sum of any q eigenvalues of (h_{jk}) is nonnegative, then*

$$\mathrm{Re}\left\{\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\big(h_{jk}Ru_{jI},u_{kI}\big)\right\}\geq\kappa\mathrm{Re}\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\big(h_{jk}u_{jI},u_{kI}\big)-C\|u\|^2,$$

and if the sum of any collection of $(n-1-q)$ eigenvalues of (h_{jk}) is nonnegative, then

$$\mathrm{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\big(h_{jj}Ru_J,u_J\big)-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\big(h_{jk}Ru_{jI},u_{kI}\big)\right\}$$

$$\geq \kappa \mathrm{Re}\left\{ \sum_{J\in\mathcal{I}_q} \sum_{j=1}^{n-1} \left(h_{jj}u_J, u_J\right) - \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(h_{jk}u_{jI}, u_{kI}\right)\right\} - C\|u\|^2.$$

Note that (h_{jk}) may be a matrix-valued function in z but may not depend on ξ.

The following lemma is the analog of Lemma 4.6 in [13].

Lemma 4.4. *Let M be as in Theorem 1.2 and φ a $(0, q)$-form supported on U so that up to a smooth term $\hat{\varphi}$ is supported in $\mathcal{C}^+$. Let*

$$(h^+_{jk}) = (c_{jk}) - \delta_{jk}\frac{1}{q}\sum_{\ell=1}^{m} c_{\ell\ell}.$$

Then

$$\mathrm{Re}\left\{ \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(h^+_{jk}T\varphi_{jI}, \varphi_{kI}\right)_\lambda\right\} \geq tA\mathrm{Re}\left\{ \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(h^+_{jk}\varphi_{jI}, \varphi_{kI}\right)_\lambda\right\}$$
$$- O(\|\varphi\|^2_\lambda) - O_t(\|\tilde{\zeta}_v \widetilde{\Psi}^0_t\varphi\|^2_0).$$

where the constant in $O(\|\varphi\|^2_\lambda)$ does not depend on t.

Proof. Observe that the eigenvalues of (h^+_{jk}) are $\mu_j - \frac{1}{q}\sum_{\ell=1}^m c_{\ell\ell}$, so the smallest possible sum of any q eigenvalues of (h^+_{jk}) is

$$\mu_1 + \cdots + \mu_q - \sum_{\ell=1}^{m} c_{\ell\ell} \geq 0.$$

With this inequality in hand, we employ the argument of Proposition 4.6 from [13] with the following changes. First, we replace c_{jk} with h^+_{jk}. Also, we replace the A with tA (for example, the sentence "By construction, $\xi_{2n-1} \geq A$ in $\mathcal{C}^+ \cdots$" gets replaced by "By construction, $\xi_{2n-1} \geq tA$ in $\mathcal{C}^+ \cdots$"). □

Observe that

$$\sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \mathrm{Re}\left(c_{jk}T\varphi_{jI}, \varphi_{kI}\right)_\lambda - \sum_{J\in\mathcal{I}_q} \sum_{j=1}^{m} \mathrm{Re}\left(c_{jj}T\varphi_J, \varphi_J\right)_\lambda$$
$$= \mathrm{Re}\left\{ \sum_{I\in\mathcal{I}_{q-1}} \sum_{j,k=1}^{n-1} \left(h^+_{jk}T\varphi_{jI}, \varphi_{kI}\right)_\lambda\right\}. \tag{11}$$

Now that we can eliminate the T terms, we turn to controlling the remaining terms.

Proposition 4.5. *Let $\varphi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b)$ be a $(0, q)$-form supported in U. Assume that λ is a q-compatible function with positivity constant B_{λ^+}. If $m < q$, choose $\lambda^+ = t\lambda$ and if $m > q$, choose $\lambda^+ = -t\lambda$. Then there exists a constant C that is independent of B_{λ^+} so that*

$$Q_{b,+}(\tilde{\zeta}\Psi^+_t\varphi, \tilde{\zeta}\Psi^+_t\varphi) + C\|\tilde{\zeta}\Psi^+_t\varphi\|^2_{\lambda^+} + O_t(\|\tilde{\zeta}\widetilde{\Psi}^0_t\varphi\|^2_0) \geq tB_{\lambda^+}\|\tilde{\zeta}\Psi^+_t\varphi\|^2_{\lambda^+}.$$

Proof. Let

$$s_{jk}^{+} = \frac{1}{2}(\overline{L}_k L_j \lambda^+ + L_j \overline{L}_k \lambda^+) + \frac{1}{2}\sum_{\ell=1}^{n-1}(d_{jk}^{\ell} L_\ell \lambda^+ + \overline{d_{kj}^{\ell}}\,\overline{L}_\ell \lambda^+)$$

and

$$r_{jk}^{+} = s_{jk}^{+} - \frac{1}{q}\delta_{jk}\sum_{\ell=1}^{m} s_{\ell\ell}.$$

In this case (10) can be rewritten as

$$Q_{b,+}(\phi, \phi) = \sum_{J\in\mathcal{I}_q}\left\{\sum_{j=1}^{m}\|\overline{L}_j^{*,+}\phi_J\|_{\lambda^+}^2 + \sum_{j=m+1}^{n-1}\|\overline{L}_j\phi_J\|_{\lambda^+}^2\right\} + E(\varphi)$$
$$+ \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\big((r_{jk}^{+} + h_{jk}^{+}T)\phi_{jI}, \phi_{kI}\big)_{\lambda}^{+}.$$

As noted in [12, 13], one can check that if $L = \sum_{j=1}^{n-1}\xi_j L_j$ (where ξ_j is constant), then

$$\left\langle\frac{1}{2}\big(\partial_b\bar{\partial}_b\lambda^+ - \bar{\partial}_b\partial_b\lambda^+\big), L\wedge\overline{L}\right\rangle = \sum_{j,k=1}^{n-1} s_{jk}^{+}\xi_j\bar{\xi}_k.$$

This means that $s_{jk}^{+} = \Theta_{jk}^{+} - \frac{1}{2}\nu(\lambda^+)c_{jk}$. Thus, if

$$\Gamma_{jk}^{\lambda^+} = \Theta_{jk}^{\lambda^+} - \frac{1}{q}\delta_{jk}\sum_{\ell=1}^{m}\Theta_{\ell\ell}^{\lambda^+}$$

then

$$Q_{b,+}(\phi, \phi) = \sum_{J\in\mathcal{I}_q}\left\{\sum_{j=1}^{m}\|\overline{L}_j^{*,+}\phi_J\|_{\lambda^+}^2 + \sum_{j=m+1}^{n-1}\|\overline{L}_j\phi_J\|_{\lambda^+}^2\right\} + E(\varphi)$$
$$+ \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\left(\left(\Gamma_{jk}^{\lambda^+} + h_{jk}^{+}\left(T - \frac{1}{2}\nu(\lambda^+)\right)\right)\phi_{jI}, \phi_{kI}\right)_{\lambda^+}.$$

Next, we replace ϕ with $\tilde{\zeta}\Psi_t^+\varphi$. Since $\operatorname{supp}\tilde{\zeta}\subset U'$, the Fourier transform of $\tilde{\zeta}\Psi_t^+\varphi$ is supported in $\mathscr{C}^+$ up to a smooth smooth term, we can use Lemma 4.4 to control the T terms. Therefore, from (10) and the form of $E(\varphi)$, we have that

$$Q_{b,+}(\tilde{\zeta}\Psi_t^+\varphi, \tilde{\zeta}\Psi_t^+\varphi) \geq (1-\epsilon)\sum_{J\in\mathcal{I}_q}\left\{\sum_{j=1}^{m}\|\overline{L}_j^{*,+}\tilde{\zeta}\Psi_t^+\varphi_J\|_{\lambda^+}^2 + \sum_{j=m+1}^{n-1}\|\overline{L}_j\tilde{\zeta}\Psi_t^+\varphi_J\|_{\lambda^+}^2\right\}$$
$$+ \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\left(\left(\Gamma_{jk}^{\lambda^+} + h_{jk}^{+}\left(tA - \frac{1}{2}\nu(\lambda^+)\right)\right)\tilde{\zeta}\Psi_t^+\varphi_{jI}, \tilde{\zeta}\Psi_t^+\varphi_{kI}\right)_{\lambda^+}$$
$$- O(\|\tilde{\zeta}\Psi_t^+\varphi\|_0^2) - O_t(\|\tilde{\zeta}_\nu\widetilde{\Psi}_t^0\varphi\|_0^2).$$

If we choose $A \geq \frac{1}{2}|v(\lambda)|$, then $tA - \frac{1}{2}v(\lambda^+) \geq 0$. Since the sum of any q eigenvalues of (h^+_{jk}) is nonnegative, these terms are strictly positive. If $m < q$, then the sum of any q eigenvalues of Γ^{λ^+} is the sum of q eigenvalues of $t\Theta^\lambda$ minus the sum of the first m diagonal terms of $t\Theta^\lambda$. If $m > q$, the sum of any q eigenvalues of Γ^{λ^+} is the sum of the first m diagonal terms of $t\Theta^\lambda$ minus the sum of q eigenvalues of $t\Theta^\lambda$. In either case, by the q-compatibility of λ, we know that this sum is at least tB_{λ^+} where B_{λ^+} is the positivity constant of λ. By Lemma 4.2, this means that

$$Q_{b,+}(\tilde{\zeta}\Psi_t^+\varphi, \tilde{\zeta}\Psi_t^+\varphi) + C\|\tilde{\zeta}\Psi_t^+\varphi\|_0^2 + O_t(\|\tilde{\zeta}_v\widetilde{\Psi}_t^0\varphi\|_0^2) \geq tB_{\lambda^+}\|\tilde{\zeta}\Psi_t^+\varphi\|_{\lambda^+}^2. \qquad \square$$

Observe that the statement of Proposition 4.5 is independent of the choice of local coordinates $L_1, \dots, L_{n-1}$ and $m \neq q$. Hence, to handle the terms with support in $\mathscr{C}^-$, we may choose new local coordinates and a new value of m so that Definitions 2.5 and 2.7 hold with $(n-1-q)$ in place of q. We again integrate (8) by parts and compute

$$\begin{aligned} &Q_{b,\lambda}(\varphi,\varphi) \\ &= \sum_{J\in\mathcal{I}_q}\left\{\sum_{j=1}^{m}\|\overline{L}_j\varphi_J\|_\lambda^2 + \sum_{j=m+1}^{n-1}\|\overline{L}_j^{*,\lambda}\varphi_J\|_\lambda^2\right\} + E(\varphi) \\ &\quad + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\big(c_{jk}T\varphi_{jI},\varphi_{kI}\big)_\lambda - \sum_{J\in\mathcal{I}_q}\sum_{j=m+1}^{n-1}\mathrm{Re}\big(c_{jj}T\varphi_J,\varphi_J\big)_\lambda \\ &\quad + \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\left\{\frac{1}{2}\big((\overline{L}_jL_k\lambda + L_j\overline{L}_k\lambda)\varphi_{jI},\varphi_{kI}\big)_\lambda + \frac{1}{2}\sum_{\ell=1}^{n-1}\big((d_{jk}^\ell L_\ell\lambda + \overline{d_{jk}^\ell}\overline{L}_\ell\lambda)\varphi_{jI},\varphi_{kI}\big)_\lambda\right\} \\ &\quad - \sum_{J\in\mathcal{I}_q}\sum_{j=m+1}^{n-1}\left\{\frac{1}{2}\big((\overline{L}_jL_j\lambda + L_j\overline{L}_j\lambda)\varphi_J,\varphi_J\big)_\lambda + \frac{1}{2}\sum_{\ell=1}^{n-1}\big((d_{jj}^\ell L_\ell\lambda + \overline{d_{jj}^\ell}\overline{L}_\ell\lambda)\varphi_J,\varphi_J\big)_\lambda\right\}. \end{aligned} \tag{12}$$

By the argument of Lemma 4.4, we can also establish the following:

Lemma 4.6. *Let M be as in Theorem 1.2 and φ be a $(0,q)$-form supported on U so that up to a smooth term, $\hat{\varphi}$ is supported in $\mathscr{C}^-$. Let*

$$(h^-_{jk}) = (c_{jk}) - \delta_{jk}\frac{1}{n-1-q}\sum_{\ell=1}^{m}c_{\ell\ell}.$$

Then

$$\begin{aligned} &\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\big(h^-_{jj}(-T)\varphi_J,\varphi_J\big)_\lambda - \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\big(h^-_{jk}(-T)\varphi_{jI},\varphi_{kI}\big)_\lambda \\ &\quad \geq tA\left(\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\big(h^-_{jj}\varphi_J,\varphi_J\big)_\lambda - \sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\big(h^-_{jk}\varphi_{jI},\varphi_{kI}\big)_\lambda\right) + O(\|\varphi\|_\lambda^2) + O_t(\|\tilde{\zeta}_v\widetilde{\Psi}_t^0\varphi\|_0^2). \end{aligned}$$

In a similar fashion to (11), we have the equality

$$\sum_{J\in\mathcal{I}_q}\sum_{j=m+1}^{n-1}\mathrm{Re}\big(c_{jj}T\varphi_J,\varphi_J\big)_\lambda-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\mathrm{Re}\big(c_{jk}T\varphi_{jI},\varphi_{kI}\big)_\lambda$$
$$=\mathrm{Re}\left\{\sum_{J\in\mathcal{I}_q}\sum_{j=1}^{n-1}\big(h^-_{jj}T\varphi_J,\varphi_J\big)_\lambda-\sum_{I\in\mathcal{I}_{q-1}}\sum_{j,k=1}^{n-1}\big(h^-_{jk}T\varphi_{jI},\varphi_{kI}\big)_\lambda\right\}. \tag{13}$$

Applying these to the proof of Proposition 4.5, we obtain

Proposition 4.7. *Let $\varphi\in\mathrm{Dom}(\bar\partial_b)\cap\mathrm{Dom}(\bar\partial_b^*)$ be a $(0,q)$-form supported in U. Assume that λ is an $(n-1-q)$-compatible function with positivity constant B_{λ^-}. If $m>n-1-q$, choose $\lambda^-=t\lambda$ and if $m<n-1-q$, choose $\lambda^-=-t\lambda$. Then there exists a constant C that is independent of B_{λ^-} so that*

$$Q_{b,-}(\tilde\zeta\Psi^-_t\varphi,\tilde\zeta\Psi^-_t\varphi)+C\|\tilde\zeta\Psi^-_t\varphi\|^2_{\lambda^-}+O_t(\|\tilde\zeta\widetilde\Psi^0_t\varphi\|^2_0)\ge tB_{\lambda^-}\|\tilde\zeta\Psi^-_t\varphi\|^2_{\lambda^-}.$$

We are now ready to prove the basic estimate, Proposition 4.1.

Proof [Proposition 4.1]. From (5), there exist constants K, $K_\pm$, and K' so that

$$KQ_{b,\pm}(\varphi,\varphi)+K_\pm\sum_\nu\|\tilde\zeta_\nu\widetilde\Psi^0_{\nu,t}\zeta_\nu\varphi^\nu\|^2_0+K'\|\varphi\|^2_0+O_\pm(\|\varphi\|^2_{-1})$$
$$\ge\sum_\nu\Big[Q_{b,+}(\tilde\zeta_\nu\Psi^+_{\nu,t}\zeta_\nu\varphi^\nu,\tilde\zeta_\nu\Psi^+_{\nu,t}\zeta_\nu\varphi^\nu)+Q_{b,-}(\tilde\zeta_\nu\Psi^-_{\nu,t}\zeta_\nu\varphi^\nu,\tilde\zeta_\nu\Psi^-_{\nu,t}\zeta_\nu\varphi^\nu)\Big].$$

From Propositions 4.5 and 4.7 it follows that by increasing the size of K, $K_\pm$, and K'

$$KQ_{b,\pm}(\varphi,\varphi)+K_\pm\sum_\nu\|\tilde\zeta_\nu\widetilde\Psi^0_{\nu,t}\zeta_\nu\varphi^\nu\|^2_0+K'\|\varphi\|^2_0+O_\pm(\|\varphi\|^2_{-1})\ge tB_\pm\|\varphi\|^2_0$$

where $B_\pm=\min\{B_{\lambda^-},B_{\lambda^+}\}$. □

4.2. *A Sobolev Estimate in the "Elliptic Directions"*

For forms whose Fourier transforms are supported up to a smooth term in $\mathscr{C}^0$, we have better estimates. The following results are in [12, 13].

Lemma 4.8. *Let φ be a $(0,1)$-form supported in U_ν for some ν such that up to a smooth term, $\hat\varphi$ is supported in $\widetilde{\mathscr{C}}^\nu_0$. There exist positive constants $C>1$ and $C_1>0$ so that*

$$CQ_{b,\pm}(\varphi,H_\pm\varphi)+C_1\|\varphi\|^2_0\ge\|\varphi\|^2_1.$$

The proof in [12] also holds at level $(0,q)$.

We can use Lemma 4.8 to control terms of the form $\|\tilde\zeta_\nu\Psi^0_{\nu,t}\zeta_\nu\varphi^\nu\|^2_0$.

Proposition 4.9. *For any $\epsilon>0$, there exists $C_{\epsilon,\pm}>0$ so that*

$$\|\tilde\zeta_\nu\Psi^0_{\nu,t}\zeta_\nu\varphi^\nu\|^2_0\le\epsilon Q_{b,\pm}(\varphi^\nu,\varphi^\nu)+C_{\epsilon,\pm}\|\varphi^\nu\|^2_{-1}.$$

See [13] for a proof of this proposition.

5. Regularity Theory for $\bar{\partial}_b$

5.1. *Closed Range for* $\Box_{b,\pm}$.

For $1 \le q \le n-2$, let

$$\begin{aligned}\mathscr{H}^q_\pm &= \{\varphi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b) : \bar{\partial}_b\varphi = 0, \bar{\partial}^*_{b,\pm}\varphi = 0\}\\ &= \{\varphi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b) : Q_{b,\pm}(\varphi,\varphi) = 0\}\end{aligned}$$

be the space of $\pm$-harmonic $(0, q)$-forms.

Lemma 5.1. *Let M^{2n-1} be a smooth, embedded CR-manifold of hypersurface type that admits a q-compatible function λ^+ and an $(n-1-q)$-compatible function λ^-. If $t > 0$ is suitably large and $1 \le q \le n-2$, then*

(i) *$\mathscr{H}^q_\pm$ is finite dimensional*;
(ii) *There exists C that does not depend on λ^+ and λ^- so that for all $(0, q)$-forms $\varphi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_b)$ satisfying $\varphi \perp \mathscr{H}^q_\pm$ (with respect to $\langle \cdot, \cdot \rangle_\pm$) we have*

$$|||\varphi|||^2_\pm \le C Q_{b,\pm}(\varphi, \varphi). \tag{14}$$

Proof. For $\varphi \in \mathscr{H}_\pm$, we can use Proposition 4.1 with t suitably large (to absorb terms) so that

$$tB_\pm |||\varphi|||^2_\pm \le C_\pm\Big(\sum_\nu \|\tilde{\zeta}_\nu \Psi^0_{\nu,t}\zeta_\mu\varphi^\nu\|^2_0 + \|\varphi\|^2_{-1}\Big).$$

Also, by Proposition 4.9,

$$\sum_\nu \|\tilde{\zeta}_\nu \Psi^0_{\nu,t}\zeta_\mu\varphi^\nu\|^2_0 \le C_\pm \|\varphi\|^2_{-1}.$$

since $Q_{b,\pm}(\varphi,\varphi) = 0$. Therefore the unit ball in $\mathscr{H}_\pm \cap L^2(M)$ is compact, and hence $\mathscr{H}_\pm$ is finite dimensional.

Assume that (14) fails. Then there exists $\varphi_k \perp \mathscr{H}_\pm$ with $|||\varphi_k|||_\pm = 1$ so that

$$|||\varphi_k|||^2_\pm \ge kQ_{b,\pm}(\varphi_k, \varphi_k). \tag{15}$$

For k suitably large, we can use Proposition 4.1 and the above argument to absorb $Q_{b,\pm}(\varphi_k, \varphi_k)$ by $B_\pm |||\varphi_k|||_\pm$ to get:

$$|||\varphi_k|||^2_\pm \le C_\pm \|\varphi_k\|^2_{-1}. \tag{16}$$

Since $L^2(M)$ is compact in $H^{-1}(M)$, there exists a subsequence φ_{k_j} that converges in $H^{-1}(M)$. However, (16) forces φ_{k_j} to converge in $L^2(M)$ as well. Although the norm $(Q_{b,\pm}(\cdot,\cdot) + |||\cdot|||^2_\pm)^{1/2}$ dominates the $L^2(M)$-norm, (15) applied to φ_{j_k} shows that φ_{j_k} converges in the $(Q_{b,\pm}(\cdot,\cdot) + |||\cdot|||^2_\pm)^{1/2}$ norm as well. The limit φ satisfies $|||\varphi|||_\pm = 1$ and $\varphi \perp \mathscr{H}_\pm$. However, a consequence of (15) is that $\varphi \in \mathscr{H}_\pm$. This is a contradiction and (14) holds. $\Box$

Let

$$^{\perp}\mathscr{H}^q_{\pm} = \{\varphi \in L^2_{0,q}(M) : \langle \varphi, \phi \rangle_{\pm} = 0, \text{ for all } \phi \in \mathscr{H}^q_{\pm}\}.$$

On $^{\perp}\mathscr{H}^q_{\pm}$, define

$$\Box_{b,\pm} = \bar{\partial}_b \bar{\partial}^*_{b,\pm} + \bar{\partial}^*_{b,\pm}\bar{\partial}_b.$$

Since $\bar{\partial}^*_{b,\pm} = H_{\pm}\bar{\partial}^*_b + [\bar{\partial}^*_b, H_{\pm}]$, $\text{Dom}(\bar{\partial}^*_{b,\pm}) = \text{Dom}(\bar{\partial}^*_b)$. This causes

$$\text{Dom}(\Box_{b,\pm}) = \{\varphi \in L^2_{0,q}(M) : \varphi \in \text{Dom}(\bar{\partial}_b) \cap \text{Dom}(\bar{\partial}^*_b),\ \bar{\partial}_b\varphi \in \text{Dom}(\bar{\partial}^*_b), \text{ and } \bar{\partial}^*_b\varphi \in \text{Dom}(\bar{\partial}_b)\}.$$

6. Proof of Theorem 1.2

6.1. *Closed Range in L^2*

From Remark 2.8, we know that $|z|^2$ is a q-compatible functions with a positivity constant of 1. Thus, for suitably large t, the space of harmonic $(0, q)$-forms $\mathscr{H}^q_t := \mathscr{H}^q_{\pm}$ is finite dimensional. Moreover, if we use $\langle \cdot, \cdot \rangle_t$ for $\langle \cdot, \cdot \rangle_{\pm}$ and $Q_{b,t}$ for $Q_{b,\pm}$, then for $\varphi \perp \mathscr{H}^q_t$ (with respect to $\langle \cdot, \cdot \rangle_t$)

$$|||\varphi|||^2_t \le C Q_{b,t}(\varphi, \varphi). \tag{17}$$

From Hörmander [8, Theorem 1.1.2], (17) is equivalent to the closed range of $\bar{\partial}_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ and $\bar{\partial}^*_{b,t} : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ where both operators are defined with respect to $\langle \cdot, \cdot \rangle_t$. By Hörmander [8, Theorem 1.1.1], this means that $\bar{\partial}^*_{b,t} : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ and $\bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ also have closed range. Thus, the Kohn Laplacian $\Box_{b,t}$ on $(0, q)$-forms also has closed range and $G_{q,t}$ exists and is a continuous operator on $L^2_{0,q}(M)$.

6.2. *Hodge Theory and the Canonical Solutions Operators*

We now prove the existence of a Hodge decomposition and the existence of the canonical solution operators. Unlike the standard computations for the $\bar{\partial}$-Neumann operators and complex Green operators in the pseudoconvex case, we only have the existence of the complex Green operator $G_{q,t}$ at a fixed level q and not for all $1 \le q \le n-1$. (hence, we cannot commute $G_{q,t}$ with either $\bar{\partial}_b$ or $\bar{\partial}^*_{b,t}$). If H^q_t is the projection of $L^2_{0,q}(M)$ onto $\mathscr{H}^q_t = \text{null}(\bar{\partial}_b) \cap \text{null}(\bar{\partial}^*_{b,t}) = \{\varphi \in L^2_{0,q}(M) \cap \text{Dom}(\bar{\partial}_b) \cap \text{Dom}(\bar{\partial}^*_{b,t}) : Q_{b,t}(\varphi, \varphi) = 0\}$, then we know

$$\varphi = \bar{\partial}_b\bar{\partial}^*_{b,t}G_{q,t}\varphi + \bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi + H^q_t\varphi.$$

We now find the canonical solution operators. Let φ be a $\bar{\partial}_b$-closed $(0, q)$-form that is orthogonal to $\mathscr{H}^q_t$. Then $H^q_t\varphi = 0$, so

$$\varphi = \bar{\partial}_b\bar{\partial}^*_{b,t}G_{q,t}\varphi + \bar{\partial}^*_{b,t}\bar{\partial}_b G_{q,t}\varphi.$$

We claim that $\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi = 0$. Following [12], we note that

$$0 = \bar\partial_b\varphi = \bar\partial_b\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi,$$

so

$$0 = \langle \bar\partial_b\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi, \bar\partial_b G_{q,t}\varphi\rangle_t = \|\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi\|_t^2.$$

Thus, $\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi = 0$ and the canonical solution operator to $\bar\partial_b$ is given by $\bar\partial^*_{b,t}G_{q,t}$. A similar argument shows that the canonical solution operator for $\bar\partial^*_{b,t}$ is given by $\bar\partial_b G_{q,t}$.

In this paragraph, we will assume that all forms are perpendicular to $\mathscr{H}^q_t$. For $\varphi \in \mathrm{Dom}(\Box_{b,t})$, it follows that

$$\varphi = G_{q,t}\Box_{b,t}\varphi = \Box_{b,t}G_{q,t}\varphi.$$

We will show that

$$\bar\partial_b\bar\partial^*_{b,t}G_{q,t} = G_{q,t}\bar\partial_b\bar\partial^*_{b,t} \quad\text{and}\quad \bar\partial^*_{b,t}\bar\partial_b G_{q,t} = G_{q,t}\bar\partial^*_{b,t}\bar\partial_b. \tag{18}$$

Observe that

$$\bar\partial_b\alpha = 0 \Longrightarrow \alpha = \bar\partial_b\bar\partial^*_{b,t}G_{q,t}\alpha = G_{q,t}\bar\partial_b\bar\partial^*_{b,t}\alpha \tag{19}$$

and

$$\bar\partial^*_{b,t}\beta = 0 \Longrightarrow \beta = \bar\partial^*_{b,t}\bar\partial_b G_{q,t}\beta = G_{q,t}\bar\partial^*_{b,t}\bar\partial_b\beta. \tag{20}$$

Next, we claim that

$$\bar\partial_b\varphi = 0 \Longrightarrow \bar\partial_b G_q\varphi = 0 \tag{21}$$

and

$$\bar\partial^*_{b,t}\varphi = 0 \Longrightarrow \bar\partial^*_{b,t}G_q\varphi = 0. \tag{22}$$

Indeed, we have that $\varphi \perp \mathscr{H}^q_t$, so $\varphi = \bar\partial_b\bar\partial^*_{b,t}G_{q,t}\varphi + \bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi$. Since Range $\bar\partial^*_{b,t} \perp \mathrm{null}\,\bar\partial_b$, $\bar\partial_b\varphi = 0$ implies that $\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi = 0$. Since $\mathrm{Range}(\bar\partial_b) \perp \mathrm{null}(\bar\partial^*_{b,t})$, $\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi = 0$ implies $\bar\partial_b G_{q,t}\varphi = 0$, as desired. A similar argument shows (22). To show (18), observe that we can write $\varphi = \alpha + \beta$ where $\bar\partial_b\alpha = 0$ and $\bar\partial^*_{b,t}\beta = 0$. Thus, by (19) and (22),

$$\bar\partial_b\bar\partial^*_{b,t}G_{q,t}\varphi = \bar\partial_b\bar\partial^*_{b,t}G_{q,t}(\alpha+\beta) = \bar\partial_b\bar\partial^*_{b,t}G_{q,t}\alpha = G_{q,t}\bar\partial_b\bar\partial^*_{b,t}\alpha = G_{q,t}\bar\partial_b\bar\partial^*_{b,t}\varphi.$$

A similar argument with (20) and (21) proves that $\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi = G_{q,t}\bar\partial^*_{b,t}\bar\partial_b\varphi$, finishing the proof of (18).

6.3. Closed Range of $\bar{\partial}_b : H^s_{0,q}(M) \to H^s_{0,q+1}(M)$ and $\bar{\partial}^*_{b,t} : H^s_{0,q}(M) \to H^s_{0,q-1}(M)$

We start with an argument to show closed range of $\bar{\partial}_b : H^s_{0,q}(M) \to H^s_{0,q+1}(M)$ and $\bar{\partial}^*_{b,t} : H^s_{0,q}(M) \to H^s_{0,q-1}(M)$. Combining Proposition 4.1 and Lemma 4.8, if t is sufficiently large, then

$$\begin{aligned}|||\Lambda^s\varphi|||_t^2 &\le \frac{C}{t}\big(|||\bar{\partial}_b\Lambda^s\varphi|||_t^2 + |||\bar{\partial}^*_{b,t}\Lambda^s\varphi|||_t^2\big) + C_t\|u\|_{s-1}^2\\ &\le \frac{C}{t}\big(|||\Lambda^s\bar{\partial}_b\varphi|||_t^2 + |||\Lambda^s\bar{\partial}^*_{b,t}\varphi|||_t^2 + |||[\bar{\partial}_b, \Lambda^s]\varphi|||_t^2 + |||[\bar{\partial}^*_{b,t}, \Lambda^s]\varphi|||_t^2\big) + C_t\|\varphi\|_{s-1}^2.\end{aligned}$$

As a consequence of Lemma 3.3, $[\bar{\partial}^*_{b,t}, \Lambda^s] = P_s + tP_{s-1}$ where P_s and P_{s-1} are pseudodifferential operators of order s and $s-1$, respectively. Additionally, $[\bar{\partial}_b, \Lambda^s]$ is a pseudodifferential operator of order s. Consequently,

$$|||\Lambda^s\varphi|||_t^2 \le \frac{C}{t}\big(|||\Lambda^s\bar{\partial}_b\varphi|||_t^2 + |||\Lambda^s\bar{\partial}^*_{b,t}\varphi|||_t^2 + |||\Lambda^s\varphi|||_t^2\big) + C_t\|\varphi\|_{s-1}^2.$$

Choosing t large enough and $\varphi \in H^s_{0,q}(M)$ allows us to absorb terms to prove

$$\begin{aligned}\|\varphi\|_s^2 = \|\Lambda^s\varphi\|_0^2 \le C_t|||\Lambda^s\varphi|||_t^2 &\le C_t\big(|||\Lambda^s\bar{\partial}_b\varphi|||_t^2 + |||\Lambda^s\bar{\partial}^*_{b,t}\varphi|||_t^2 + \|\varphi\|_{s-1}^2\big)\\ &\le C_t\big(\|\bar{\partial}_b\varphi\|_s^2 + \|\bar{\partial}^*_{b,t}\varphi\|_s^2 + \|\varphi\|_{s-1}^2\big).\end{aligned}$$

Thus, $\bar{\partial}_b : H^s_{0,q}(M) \to H^s_{0,q+1}(M)$ and $\bar{\partial}^*_{b,t} : H^s_{0,q}(M) \to H^s_{0,q-1}(M)$ have closed range.

6.4. Continuity of the Complex Green Operator in $H^s_{0,q}(M)$

We now turn to the harder problem of showing continuity of the complex Green operator $G^\delta_{q,t}$ in $H^s_{0,q}(M)$, $s > 0$. We use an elliptic regularization argument. Let $Q^\delta_{b,t}(\cdot,\cdot)$ be the quadratic form on $H^1_{0,q}(M)$ defined by

$$Q^\delta_{b,t}(u, v) = Q_{b,t}(u, v) + \delta Q_{d_b}(u, v)$$

where Q_{d_b} is the hermitian inner product associated to the de Rham exterior derivative d_b, i.e., $Q_{d_b}(u, v) = \langle d_b u, d_b v\rangle_t + \langle d_b^* u, d_b^* v\rangle_t$. The inner product Q_{d_b} has form domain $H^1_{0,q}(M)$. Consequently, $Q^\delta_{b,t}$ gives rise to a unique, self-adjoint, elliptic operator $\Box^\delta_{b,t}$ with inverse $G^\delta_{q,t}$.

From Proposition 4.1 and Lemma 4.8, if t is large enough, then for $\varphi \in \mathrm{Dom}(\bar{\partial}_b) \cap \mathrm{Dom}(\bar{\partial}^*_{b,t})$, we have the estimate

$$|||\varphi|||_t^2 \le \frac{K}{t}Q_{b,t}(\varphi, \varphi) + C_t\|\varphi\|_{-1}^2. \tag{23}$$

Now let $\varphi \in H^s_{0,q}(\Omega)$. Since $\Box^\delta_{b,t}$ is elliptic, $G^\delta_{q,t}\varphi \in H^{s+2}_{0,q}(M)$. Then

$$\|G^\delta_{q,t}\varphi\|_s^2 = \|\Lambda^s G^\delta_{q,t}\varphi\|_0^2 \le C_t|||\Lambda^s G^\delta_{q,t}\varphi|||_t^2. \tag{24}$$

We now concentrate on finding a bound for $|||\Lambda^s G^\delta_{q,t}\varphi|||_t^2$ that is independent of δ. By (23),

$$|||\Lambda^s G^\delta_{q,t}\varphi|||_t^2 \le \frac{K}{t} Q_{b,t}(\Lambda^s G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi) + C_{t,s}\|G^\delta_{q,t}\varphi\|_{s-1}^2. \tag{25}$$

Observe that if $(\Lambda^s)^{*,t}$ is the adjoint of Λ^s under the inner product $\langle\cdot,\cdot\rangle_t$, then

$$\langle\Lambda^s u, v\rangle_t = (u, \Lambda^s H_t^{-1}v)_0 = \langle u, H_t\Lambda^s H_t^{-1}v\rangle_t = \langle u, (\Lambda^s + [H_t, \Lambda^s]H_t^{-1})v\rangle_t$$

implies that $(\Lambda^s)^{*,t} = \Lambda^s + [H_t, \Lambda^s]H_t^{-1}$. Therefore, it is a standard consequence of [10, Lemma 3.1] (or [7, Lemma 2.4.2]) that

$$\begin{aligned} Q_{b,t}(\Lambda^s G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi) &\le Q^\delta_{b,t}(\Lambda^s G^\delta_{q,t}\varphi, \Lambda^s G^\delta_{q,t}\varphi) \\ &\le |\langle\Lambda^s\varphi, \Lambda^s G^\delta_{q,t}\varphi\rangle_t| + C\|G^\delta_{q,t}\varphi\|_s^2 + C_{t,s}\|G^\delta_{q,t}\varphi\|_{s-1}^2 \\ &\le |||\Lambda^s\varphi|||_t |||\Lambda^s G^\delta_{q,t}\varphi|||_t + \|G^\delta_{q,t}\varphi\|_{s-1}^2 \\ &\le K_t\|\varphi\|_s^2 + C|||\Lambda^s G^\delta_{q,t}\varphi|||_t^2 + C_{t,s}\|G^\delta_{q,t}\varphi\|_{s-1}^2 \end{aligned} \tag{26}$$

where $C > 0$ does not depend on δ or t.

Plugging (26) into (25), we see that

$$|||\Lambda^s G^\delta_{q,t}\varphi|||_t^2 \le \frac{K}{t}\Big(K_t\|\varphi\|_s^2 + C|||\Lambda^s G^\delta_{q,t}\varphi|||_t^2\Big) + C_{t,s}\|G^\delta_{q,t}\varphi\|_{s-1}^2.$$

If t is sufficiently large, then it follows that

$$|||\Lambda^s G^\delta_{q,t}\varphi|||_t^2 \le K_t\|\varphi\|_s^2 + C_{t,s}\|G^\delta_{q,t}\varphi\|_{s-1}^2 \tag{27}$$

since $|||\Lambda^s G^\delta_{q,t}\varphi|||_t^2 < \infty$ (recall that $G^\delta_{q,t}\varphi \in H^{s+2}_{0,q}(M)$). Plugging (27) into (24), we have the bound

$$\|G^\delta_{q,t}\varphi\|_s^2 \le K_t\|\varphi\|_s^2 + C_{t,s}\|G^\delta_{q,t}\varphi\|_{s-1}^2. \tag{28}$$

We now turn to letting $\delta \to 0$. Observe that K_t and $C_{t.s}$ are independent of δ. We have shown that if $\varphi \in H^s_{0,q}(M)$, then $\{G^\delta_{q,t}\varphi : 0 < \delta < 1\}$ is bounded in $H^s_{0,q}(M)$. Thus, there exists a sequence $\delta_k \to 0$ and $\tilde{u} \in H^s_{0,q}(M)$ so that $G^{\delta_k}_{q,t}u \to \tilde{u}$ weakly in $H^s_{0,q}(M)$. Consequently, if $v \in H^{s+2}_{0,q}(M)$, then

$$\lim_{k\to\infty} Q^{\delta_k}_{b,t}(G^{\delta_k}_{q,t}u, v) = Q_{b,t}(\tilde{u}, v).$$

However,

$$Q^{\delta_k}_{b,t}(G^{\delta_k}_{q,t}u, v) = (u, v) = Q_{b,t}(G_{q,t}u, v),$$

so $G_{q,t}u = \tilde{u}$ and (28) is satisfied with $\delta = 0$. Thus, $G_{q,t}$ is a continuous operator on $H^s_{0,q}(M)$.

6.5. *Continuity of the Canonical Solution Operators in $H^s_{0,q}(M)$*

Continuity of $\bar\partial_b G_{q,t}$ and $\bar\partial^*_{b,t} G_{q,t}$ will follow from the continuity of $G_{q,t}$. Unfortunately, we cannot apply Proposition 4.1 to either $\bar\partial_b G_{q,t}\varphi$ or $\bar\partial^*_{b,t} G_{q,t}\varphi$ because neither are $(0, q)$-forms. Instead, we estimate directly:

$$\begin{aligned}
\|\bar\partial_b G_{q,t}\varphi\|_s^2 + \|\bar\partial^*_{b,t} G_{q,t}\varphi\|_s^2 &\le C_t(|\!|\!|\Lambda^s\bar\partial_b G_{q,t}\varphi|\!|\!|_t^2 + |\!|\!|\Lambda^s\bar\partial^*_{b,t} G_{q,t}\varphi|\!|\!|_t^2)\\
&= C_t\Big(\langle\Lambda^s\varphi, \Lambda^s G_{q,t}\varphi\rangle_t + \langle\Lambda^s\bar\partial_b G_{q,t}\varphi, [\Lambda^s, \bar\partial_b]G_{q,t}\varphi\rangle_t + \langle[\bar\partial^*_{b,t}, \Lambda^s]\bar\partial_b G_{q,t}\varphi, \Lambda^s G_{q,t}\varphi\rangle_t\\
&\quad + \langle\Lambda^s\bar\partial^*_{b,t} G_{q,t}\varphi, [\Lambda^s, \bar\partial^*_{b,t}]G_{q,t}\varphi\rangle_t + \langle[\bar\partial_b, \Lambda^s]\bar\partial^*_{b,t} G_{q,t}\varphi, \Lambda^s G_{q,t}\varphi\rangle_t\Big)\\
&\le C_{t,s}(\|\varphi\|_s^2 + \|G_{q,t}\varphi\|_s^2) \le C_{t,s}\|\varphi\|_s^2.
\end{aligned}$$

6.6. *The Szegö Projection $S_{q,t}$*

The Szegö projection $S_{q,t}$ is the projection of $L^2_{0,q}(M)$ onto $\ker\bar\partial_b$. We claim that

$$S_{q,t} = I - \bar\partial^*_{b,t}\bar\partial_b G_{q,t} = I - G_{q,t}\bar\partial^*_{b,t}\bar\partial_b.$$

The second equality follows from (18). Observe that if $\varphi \in \mathrm{null}(\bar\partial_b)$, then $(I - G_{q,t}\bar\partial^*_{b,t}\bar\partial_b)\varphi = \varphi$, as desired. If $\varphi \perp \mathrm{null}(\bar\partial_b)$, then $\varphi \perp \mathscr{H}^q_t$, so $\varphi = \bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi + \bar\partial_b\bar\partial^*_{b,t} G_{q,t}\varphi$. We claim that $\varphi = \bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi$. Let $u = \bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi$. Then u is the canonical solution to $\bar\partial_b u = \bar\partial_b\varphi$, so $\bar\partial_b(\varphi - u) = 0$. However, $\varphi \perp \mathrm{null}(\bar\partial_b)$, so $u = \varphi$, and $0 = \varphi - u = (I - \bar\partial^*_{b,t}\bar\partial_b G_{q,t})\varphi$, as desired.

Proposition 6.1. *Let M be as in Theorem* 1.2. *If $t \ge T_s$, then the Szegö kernel $S_{q,t}$ is continuous on $H^s_{0,q}(M)$.*

Proof. This argument uses ideas from [3]. Given $\varphi \in L^2_{0,q}(M)$, we know that $\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi \in L^2_{0,q}(M)$, but we have no quantitative bound. However,

$$|\!|\!|\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi|\!|\!|_t^2 = \langle\bar\partial_b\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi, \bar\partial_b G_{q,t}\varphi\rangle_t = \langle\bar\partial_b\varphi, \bar\partial_b G_{q,t}\varphi\rangle_t \le |\!|\!|\varphi|\!|\!|_t |\!|\!|\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi|\!|\!|_t.$$

This proves continuity in $L^2_{0,q}(M)$.

Now let $s > 0$. It suffices to show

$$|\!|\!|\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G_{q,t}\varphi|\!|\!|_t^2 \le C_{s,t}|\!|\!|\Lambda^s\varphi|\!|\!|_t^2. \tag{29}$$

We cannot simply integrate by parts as in the L^2-case because we do not know if $\Lambda^s\bar\partial^*_{b,t}\bar\partial_b S_{q,t}\varphi$ is finite. As above, we can avoid this issue by an elliptic regularity argument. Using the operators $G^\delta_{q,t}$ from §6.4, we have (if δ is small enough)

$$\begin{aligned}
|\!|\!|\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G^\delta_{q,t}\varphi|\!|\!|_t^2 &= \langle\Lambda^s\bar\partial_b\bar\partial^*_{b,t}\bar\partial_b G^\delta_{q,t}\varphi, \Lambda^s\bar\partial_b G^\delta_{q,t}\varphi\rangle_t + \langle[\bar\partial_b, \Lambda^s]\bar\partial^*_{b,t}\bar\partial_b G^\delta_{q,t}\varphi, \Lambda^s\bar\partial_b G^\delta_{q,t}\varphi\rangle_t\\
&\quad + \langle\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G^\delta_{q,t}\varphi, [\Lambda^s, \bar\partial^*_{b,t}]\bar\partial_b G^\delta_{q,t}\varphi\rangle_t\\
&\le C_{s,t}(|\!|\!|\Lambda^s\varphi|\!|\!|_t + |\!|\!|\Lambda^s\bar\partial_b G^\delta_{q,t}\varphi|\!|\!|_t)|\!|\!|\Lambda^s\bar\partial^*_{b,t}\bar\partial_b G^\delta_{q,t}\varphi|\!|\!|_t.
\end{aligned}$$

Using that the continuity of $\bar{\partial}_b G^{\delta}_{q,t}$ in $H^s_{0,q}(M)$ is uniform in δ (for small δ), we have

$$|||\Lambda^s \bar{\partial}^*_{b,t}\bar{\partial}_b G^{\delta}_{q,t}\varphi|||_t \le C_{s,t}(|||\Lambda^s\varphi|||_t + |||\Lambda^s\bar{\partial}_b G_{q,t}\varphi|||_t) \le C_{s,t}|||\Lambda^s\varphi|||_t. \tag{30}$$

As earlier, we can take an appropriate limit as $\delta \to 0$ to establish the bound in (30) with $\delta = 0$. □

6.7. *Results for Levels* $(0, q-1)$ *and* $(0, q+1)$

We now show continuity of the canonical solution operators $G_{q,t}\bar{\partial}^*_{b,t} : H^s_{0,q+1}(M) \to H^s_{0,q}(M)$ and $G_{q,t}\bar{\partial}_b : H^s_{0,q-1}(M) \to H^s_{0,q}(M)$, and the Szegö projection $S_{q-1,t} = I - \bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b : H^s_{0,q-1}(M) \to H^s_{0,q-1}(M)$. We cannot express the Szegö kernel of $(0, q+1)$-forms in terms of $G_{q,t}$ because the only candidate is $\bar{\partial}_b G_{q,t}\bar{\partial}^*_{b,t}$, but this object annihilates t-harmonic forms (which ought to remain unchanged by $S_{q+1,t}$). Since $H^{s+1}_{0,q-1}(M)$ is dense in $H^s_{0,q-1}(M)$ and $G_{q,t}$ preserves $H^s_{0,q}(M)$, we may assume that $\varphi \in H^{s+1}_{0,q-1}(M)$. Then

$$\begin{aligned}|||\Lambda^s G_{q,t}\bar{\partial}_b\varphi|||^2_t &= \Big\langle \underbrace{\bar{\partial}^*_{b,t}G_{q,t}}_{\text{bounded in } H^s} \Lambda^s G_{q,t}\bar{\partial}_b\varphi, \Lambda^s\varphi\Big\rangle_t + \langle \Lambda^s G_{q,t}\bar{\partial}_b\varphi, [\Lambda^s, G_{q,t}\bar{\partial}_b]\varphi\rangle_t \\ &\le C_s|||\Lambda^s G_{q,t}\bar{\partial}_b\varphi|||_t|||\Lambda^s\varphi|||_t.\end{aligned}$$

The right hand side is finite since $\bar{\partial}_b\varphi \in H^s_{0,q}(M)$ by assumption. Thus, $G_{q,t}\bar{\partial}_b : H^s_{0,q-1}(M) \to H^s_{0,q}(M)$ is bounded. A similar argument shows that $G_{q,t}\bar{\partial}^*_{b,t} : H^s_{0,q+1}(M) \to H^s_{0,q}(M)$ is continuous.

For the Szegö projection, we investigate the boundedness of

$$\begin{aligned}|||\Lambda^s\bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b\varphi|||^2_t &= \langle \Lambda^s\bar{\partial}_b\bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b\varphi, \Lambda^s G_{q,t}\bar{\partial}_b\varphi\rangle_t + \langle \Lambda^s\bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b\varphi, [\Lambda^s, \bar{\partial}^*_{b,t}]G_{q,t}\bar{\partial}_b\varphi\rangle_t \\ &\quad + \langle [\bar{\partial}^*_{b,t}, \Lambda^s]\bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b\varphi, \Lambda^s G_{q,t}\bar{\partial}_b\varphi\rangle_t.\end{aligned}$$

Since $\bar{\partial}_b\varphi$ is $\bar{\partial}_b$-closed, $\bar{\partial}_b\bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b\varphi = \bar{\partial}_b\varphi$, so

$$\begin{aligned}\langle \Lambda^s\bar{\partial}_b\bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b\varphi, \Lambda^s G_{q,t}\bar{\partial}_b\varphi\rangle_t &= \langle \Lambda^s\varphi, \Lambda^s\bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b\varphi\rangle_t + \langle [\Lambda^s, \bar{\partial}_b]\varphi, \Lambda^s G_{q,t}\bar{\partial}_b\varphi\rangle_t \\ &\quad + \langle \Lambda^s\varphi, [\Lambda^s, \bar{\partial}^*_{b,t}]G_{q,t}\bar{\partial}_b\varphi\rangle_t \\ &\le C_s(|||\Lambda^s\varphi|||_t|||\Lambda^s\bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b\varphi|||_t + |||\Lambda^s\varphi|||^2_t).\end{aligned}$$

Thus, we have

$$|||\Lambda^s\bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b\varphi|||^2_t \le C_{s,t}(|||\Lambda^s\varphi|||_t|||\Lambda^s\bar{\partial}^*_{b,t}G_{q,t}\bar{\partial}_b\varphi|||_t + |||\Lambda^s\varphi|||^2_t).$$

Using a small constant/large constant argument and absorbing terms, we have the continuity of the Szegö projection in $H^s_{0,q-1}(M)$.

The continuity of the solution operator $\bar{\partial}^*_{b,t}G_{q,t}$ immediately gives closed range of $\bar{\partial}_b$ from $H^s_{0,q-1}(M)$ to $H^s_{0,q}(M)$. Similarly, the boundedness of the operator $\bar{\partial}_b G_{q,t}$ immediately gives closed range of $\bar{\partial}^*_b$ from $H^s_{0,q+1}(M)$ to $H^s_{0,q}(M)$.

6.8. Exact and Global Regularity for $\bar{\partial}_b$

In this section, we prove that if $\alpha \in C^\infty_{0,\tilde{q}+1}(M)$ satisfies $\bar{\partial}_b\alpha = 0$ and $\alpha \perp \mathscr{H}_t^{\tilde{q}}$, then there exists $u \in C^\infty_{0,\tilde{q}}(M)$ so that $\bar{\partial}_b u = \alpha$ where $\tilde{q} = q$ or $q-1$. We follow the argument in [12, Lemma 5.10]. We start by showing that if k is fixed and $s > k$, then $H^s_{0,\tilde{q}}(M) \cap \mathrm{null}(\bar{\partial}_b)$ is dense in $H^k_{0,\tilde{q}}(M) \cap \mathrm{null}(\bar{\partial}_b)$. Let $g \in H^k_{0,\tilde{q}}(M) \cap \mathrm{null}(\bar{\partial}_b)$. Since $C^\infty_{0,\tilde{q}}(M)$ is dense in $H^k_{0,\tilde{q}}(M)$, there exists a sequence $g_j \in C^\infty_{0,\tilde{q}}(M)$ so that $g_j \to g$ in $H^k_{0,\tilde{q}}(M)$. Let $t \geq T_s$ and set $\tilde{g}_j = S_{\tilde{q},t}g_j$. By the continuity of $S_{\tilde{q},t}$ in $H^s_{0,\tilde{q}}(M)$, $\tilde{g}_j \in H^s_{0,\tilde{q}}(M)$. Moreover, since $g = S_{\tilde{q},t}g$, it follows that

$$\lim_{j\to\infty} \|\tilde{g}_j - g\|_k^2 = \lim_{j\to\infty} \|S_{\tilde{q},t}(g_j - g)\|_k^2 \leq C_{k,t} \lim_{j\to\infty} \|g_j - g\|_k^2 = 0.$$

Next, since $\alpha = \bar{\partial}_b\bar{\partial}^*_{b,t}G_{\tilde{q},t}\alpha$ or $\bar{\partial}_b G_{\tilde{q},t}\bar{\partial}^*_{b,t}\alpha$ for all sufficiently large t, by choosing an appropriate sequence $t_k \to \infty$, there exists $u_k = \bar{\partial}^*_{b,t}G_{\tilde{q},t_k}\alpha$ or $G_{\tilde{q},t_k}\bar{\partial}^*_{b,t}\alpha \in H^k_{0,\tilde{q}}(M)$ so that $\bar{\partial}_b u_k = \alpha$. We will construct a sequence $\tilde{u}_k$ inductively. Let $\tilde{u}_1 = u_1$. Assume that $\tilde{u}_k$ has been defined so that $\tilde{u}_k \in H^k_{0,\tilde{q}}(M)$, $\bar{\partial}_b\tilde{u}_k = \alpha$, and $\|\tilde{u}_k - \tilde{u}_{k-1}\|_{k-1} \leq 2^{k-1}$. We will now construct $\tilde{u}_{k+1}$. Note that $\bar{\partial}_b(u_{k+1} - \tilde{u}_k) = 0$. By the density argument above, there exists $v_{k+1} \in H^{k+1}_{0,\tilde{q}}(M) \cap \mathrm{null}(\bar{\partial}_b)$ so that if $\tilde{u}_{k+1} = u_{k+1} + v_{k+1}$, then $\|\tilde{u}_{k+1} - \tilde{u}_k\|_k \leq 2^{-k}$. Finally, set

$$u = \tilde{u}_1 + \sum_{k=1}^{\infty}(\tilde{u}_{k+1} - \tilde{u}_k) = \tilde{u}_j + \sum_{k=j}^{\infty}(\tilde{u}_{k+1} - \tilde{u}_k), \quad j \in \mathbb{N}.$$

The sum telescopes and it is clear that $u \in H^j_{0,\tilde{q}}(M)$ for all $j \in \mathbb{N}$ and $\bar{\partial}_b u = \alpha$. Thus, $u \in C^\infty_{0,\tilde{q}}(M)$.

7. Proof of Theorem 1.1

From (3), we know that weighted $L^2(M)$ and $L^2(M)$ are equivalent spaces. Thus, from Theorem 1.2, we know that $\bar{\partial}_b : L^2_{0,q-1}(M) \to L^2_{0,q}(M)$ and $\bar{\partial}_b : L^2_{0,q}(M) \to L^2_{0,q+1}(M)$ have closed range. Again by Hörmander [8, Theorem 1.1.1], this proves that $\bar{\partial}_b^* : L^2_{0,q}(M) \to L^2_{0,q-1}(M)$ and $\bar{\partial}_b^* : L^2_{0,q+1}(M) \to L^2_{0,q}(M)$ have closed range. Consequently, the Kohn Laplacian $\Box_b = \bar{\partial}_b\bar{\partial}_b^* + \bar{\partial}_b^*\bar{\partial}_b$ has closed range on $L^2_{0,q}(M)$ and the remainder of the theorem follows by standard arguments. This concludes the proof of Theorem 1.1.

Remark 7.1. This is more quantitative discussion of Remark 1.3. In particular, from the proof of Theorem 1.1, we have the closed range bound for appropriate $(0, q)$-forms φ (using (3)),

$$\|\varphi\|_0^2 \leq \frac{1}{c_t}\|\varphi\|_t^2 \leq \frac{C}{c_t}\|\bar{\partial}_b\varphi\|_t^2 \leq \frac{CC_t}{c_t}\|\bar{\partial}_b\varphi\|_0^2.$$

Thus, the closed range constants for $\bar{\partial}_b$, $\bar{\partial}_b^*$, and $\Box_b$ in unweighted $L^2(M)$ depend on the size of λ^+ and λ^-.

Acknowledgements

The first author is partially supported by NSF grant DMS-1002332 and the second author is partially supported by NSF grant DMS-0855822.

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