J Geom Anal (2011) 21: 256–275
DOI 10.1007/s12220-010-9146-z

The $\Box_b$-Heat Equation on Quadric Manifolds

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Received: 4 July 2009 / Published online: 8 July 2010

Abstract In this article, we give an explicit calculation of the partial Fourier transform of the fundamental solution to the $\Box_b$-heat equation on quadric submanifolds $M \subset \mathbb{C}^n \times \mathbb{C}^m$. As a consequence, we can also compute the heat kernel associated with the weighted $\bar{\partial}$-equation in $\mathbb{C}^n$ when the weight is given by $\exp(-\phi(z,z)\cdot\lambda)$ where $\phi : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^m$ is a quadratic, sesquilinear form and $\lambda \in \mathbb{R}^m$. Our method involves the representation theory of the Lie group M and the group Fourier transform.

Keywords Quadric manifold · Lie group · Heat kernel · Heat equation · Fundamental solution · Kohn Laplacian · Heisenberg group

Mathematics Subject Classification (2000) Primary 32W30 · 33C45 · 43A80 · 35K08

1 Introduction

The purpose of this article is to present an explicit calculation of the Fourier transform of the fundamental solution of the $\Box_b$-heat equation on quadric submanifolds $M \subset \mathbb{C}^n \times \mathbb{C}^m$. A quadric submanifold can be thought of as a generalization of the

Communicated by Steve Bell.

The second author is partially funded by NSF grant DMS-0855822.

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Heisenberg group—it is a Lie group with a known representation theory [13], and the technique of using Hermite functions to compute the heat kernel, as done in [4, 12] and elsewhere, can be extended to work in this situation as well.

A consequence of our fundamental solution computation is that we can explicitly compute the heat kernel associated with the weighted $\bar{\partial}$-problem in $\mathbb{C}^n$ when the weight is given by $\exp(-\phi(z,z)\cdot\lambda)$ where $\phi:\mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^m$ is a quadratic, sesquilinear form and $\lambda\in\mathbb{R}^m$. This computation partially generalizes the results in [4]. When $m=1$ and the weight is given by $\exp(\tau P(z_1,\dots,z_n))$ where $\tau\in\mathbb{R}$, $P(z_1,\dots,z_n)=\sum_{j=1}^n p_j(z_j)$, and p_j are subharmonic, nonharmonic polynomials, Raich [14–17] has estimated the heat kernel associated with the weighted $\bar{\partial}$-problem. If, in addition, $n=1$, the weighted $\bar{\partial}$-problem and explicit construction of Bergman and Szegö kernels have been studied by a number of authors in different contexts, e.g., [1, 6, 8–11]. We also note that quadric manifolds are related to H-type groups on which Yang and Zhu have computed the heat kernel for the sub-Laplacian [20]. Additionally, although there is some overlap with the results by Calin et al. [5], their method is based on Hamilton-Jacobi theory in the spirit of Beals et al. [2, 3] and they only consider the case when ϕ is diagonal.

The remainder of the paper is organized as follows: in Sect. 2, we define our terms and state our main results. Section 3 provides the necessary background from representation theory. In Sects. 4 and 5, we apply the representation theory of M to the heat kernels and prove the main results.

2 Quadric Submanifolds and the $\Box_b$-Heat Equation

2.1 Quadric Submanifolds

Let M be the quadric submanifold in $\mathbb{C}^n\times\mathbb{C}^m$ defined by

$$M=\{(z,w)\in\mathbb{C}^n\times\mathbb{C}^m;\ \operatorname{Im} w=\phi(z,z)\}$$

where $\phi:\mathbb{C}^n\times\mathbb{C}^n\mapsto\mathbb{C}^m$ is a sesquilinear form (i.e., $\phi(z,z')=\overline{\phi(z',z)}$). For emphasis, we sometimes write M_ϕ to denote the dependence of M on the quadratic function ϕ. Note that $M_{-\phi}$ is biholomorphic to M_ϕ by the change of variables $(z,w)\mapsto(z,-w)$.

For $\lambda\in\mathbb{R}^m$, let

$$\phi^\lambda(z,z')=\phi(z,z')\cdot\lambda$$

where $\cdot$ is the ordinary dot product (without conjugation). Observe that $\phi^\lambda(z,z')$ is a sesquilinear scalar-valued form with an associated Hermitian matrix. Let $v_1^\lambda,\dots,v_n^\lambda$ be an orthonormal basis for $\mathbb{C}^n$ with

$$\phi^\lambda(v_j^\lambda,v_k^\lambda)=\delta_{jk}\mu_j(\lambda)$$

where $\mu_j(\lambda)=\mu_j^\lambda$ are the eigenvalues of the matrix associated with ϕ^λ.

2.2 Lie Group Structure

After projecting $M \subset \mathbb{C}^n \times \mathbb{C}^m$ onto $G = \mathbb{C}^n \times \mathbb{R}^m$, the Lie group structure of M is isomorphic to the following group structure on G:

$$gg' = (z,t)(z',t') = (z + z', t + t' + 2\,\mathrm{Im}\,\phi(z,z')).$$

Note that $(0,0)$ is the identity in this group structure and that the inverse of (z,t) is $(-z,-t)$.

The *right invariant* vector fields are given as follows: let $g \in G$; if X is a vector field, then we denote its value at g by $X(g)$, an element of the tangent space of M at g. Define $R_g : G \mapsto G$ by $R_g(g') = g'g$; then the right invariant vector fields, $X(g)$, are obtained by pushing forward the vectors in the tangent space at the origin via the differential of the map R_g. In particular, a vector field X is right invariant if and only if $X(g) = (R_g)_*\{X(0)\}$, where $(R_g)_*$ denotes the push forward operator. Let v be a vector in $\mathbb{C}^n \approx \mathbb{R}^{2n}$ which can be identified with the tangent space of M at the origin. Let ∂_v be the real vector field given by the directional derivative in the direction of v. Then the right invariant vector field at an arbitrary $g = (z,w) \in M$ corresponding to v is given by

$$X_v(g) = \partial_v + 2\,\mathrm{Im}\,\phi(v,z)\cdot D_t = \partial_v - 2\,\mathrm{Im}\,\phi(z,v)\cdot D_t$$

where $D_t = (\partial_{t_1},\ldots,\partial_{t_m})$, (see Sect. 1 in Peloso/Ricci [13]). Let Jv be the vector in $\mathbb{R}^{2n}$ which corresponds to iv in $\mathbb{C}^n$ (where $i = \sqrt{-1}$). The CR structure on G is then spanned by vectors of the form:

$$Z_v(g) = (1/2)(X_v - iX_{Jv}) = (1/2)(\partial_v - i\partial_{Jv}) - i\overline{\phi(z,v)}\cdot D_t$$

and

$$\overline{Z}_v(g) = (1/2)(X_v + iX_{Jv}) = (1/2)(\partial_v + i\partial_{Jv}) + i\phi(z,v)\cdot D_t.$$

Also,

$$[X_v, X_{v'}] = 4\,\mathrm{Im}\,\phi(v',v)\cdot D_t, \qquad [Z_v, Z_{v'}] = 0$$

and

$$[\overline{Z}_v, \overline{Z}_{v'}] = 0, \qquad [Z_v, \overline{Z}_{v'}] = 2i\phi(v,v')\cdot D_t.$$

We often drop the g in the vector field notation. The vector field definition of the Levi form of M is the map $v \mapsto \mathrm{proj}([Z_v, \overline{Z}_v])$, where proj stands for the projection onto the totally real part of the tangent space of M at the origin (i.e., the t-axes). From the above equation, the Levi form of M can be identified with the map $v \mapsto \phi(v,v)$, as mentioned at the beginning of this section.

Recall that for any $\lambda \in \mathbb{R}^m$, the set of vectors $v_1^\lambda,\ldots,v_n^\lambda$ is an orthonormal basis which diagonalizes $\phi^\lambda(z,z) = \phi(z,z)\cdot\lambda$. For $\lambda \in \mathbb{R}^m$, define the function $\nu(\lambda)$ by

$$\nu(\lambda) = \mathrm{rank}(\phi^\lambda).$$

The function $\nu(\lambda)$ satisfies $0 \le \nu(\lambda) \le n$ and, as in [13],

$$\{\lambda \in \mathbb{R}^m : \nu(\lambda) \equiv \max_{\tilde{\lambda}\in\mathbb{R}^m} \nu(\tilde{\lambda})\}$$

is a Zariski-open set $\Omega \subset \mathbb{R}^m$ that carries full measure, i.e., $|\mathbb{R}^m \setminus \Omega| = 0$. We identify x with $(x_1^\lambda, \dots, x_n^\lambda)$ and y with $(y_1^\lambda, \dots, y_n^\lambda)$ We also write $z = \sum_{j=1}^n (x_j^\lambda + i y_j^\lambda) v_j^\lambda$ for $z = x + iy \in \mathbb{C}^n$. Additionally, we let $z' = (z_1^\lambda, \dots, z_{\nu(\lambda)}^\lambda)$, $z'' = (z_{\nu(\lambda)+1}^\lambda, \dots, z_n^\lambda)$ and similarly for x and y.

Since the right invariant vector fields corresponding to ϕ are equal to the left invariant vector fields corresponding to $-\phi$ and $M_{-\phi}$ is biholomorphic to M_ϕ, any analysis involving right invariant vector fields yields corresponding information about the left invariant vector fields and vice versa.

2.3 $\Box_b$ Calculations

Let $v_1, \dots, v_n$ be any orthonormal basis for $\mathbb{C}^n$. Let $X_j = X_{v_j}$, $Y_j = X_{Jv_j}$, and let $Z_j = (1/2)(X_j - iY_j)$, $\overline{Z}_j = (1/2)(X_j + iY_j)$ be the right invariant vector fields defined above (which are also the left invariant vector fields for the group structure with ϕ replaced by $-\phi$). Also let dz_j and $d\overline{z}_j$ be the dual basis. A $(0,q)$-form can be expressed as $\sum_{K\in\mathcal{I}_q} \phi_K \, d\overline{z}^K$ where $\mathcal{I}_q = \{K = (k_1, \dots, k_q) : 1 \le k_1 < \cdots < k_q \le n\}$. Proposition 2.1 in [13] states that

$$\Box_b\Big(\sum_{K\in\mathcal{I}_q} \phi_K \, d\overline{z}^K\Big) = \sum_{K,L\in\mathcal{I}_q} \Box_{LK}\phi_K \, d\overline{z}^L$$

where

$$\Box_{LK} = -\delta_{LK}\mathcal{L} + M_{LK} \tag{1}$$

with the sub-Laplacian on G

$$\mathcal{L} = (1/2)\sum_{k=1}^n (\overline{Z}_k Z_k + Z_k \overline{Z}_k)$$

and

$$M_{LK} = \begin{cases} \frac{1}{2}\big(\sum_{k\in K}[Z_k, \overline{Z}_k] - \sum_{k\notin K}[Z_k, \overline{Z}_k]\big) & \text{if } K = L, \\ \epsilon(K, L)[Z_k, \overline{Z}_l] & \text{if } |K \cap L| = q-1, \\ 0 & \text{otherwise.} \end{cases}$$

Here, $\epsilon(K, L)$ is $(-1)^d$ where d is the number of elements in $K \cap L$ between the unique element $k \in K - L$ and the unique element $l \in L - K$. The above theorem is stated and proved in [13] for the left invariant vector fields. If right invariant vector fields are used, then the above theorem provides a formula for $\Box_b$ associated with $M_{-\phi}$.

For later, we record the diagonal part of $\Box_b$, i.e., $\Box_{LL}$. Using (1) with $L = K$ and the above formulas for Z_k, we obtain

$$\Box_{LL} = -\frac{1}{4}\Delta + 2\,\mathrm{Im}\left\{\sum_{k=1}^{n}\phi(z, v_k)\partial_{z_k}\right\}\cdot D_t - \sum_{k=1}^{n}\big(\phi(z, v_k)\cdot D_t\big)\big(\overline{\phi(z, v_k)}\cdot D_t\big)$$
$$+ i\left(\sum_{k\in L}\phi(v_k, v_k)\cdot D_t - \sum_{k\notin L}\phi(v_k, v_k)\cdot D_t\right) \tag{2}$$

where Δ is the usual Laplacian in the z-coordinates. For example, in the classic case of the Heisenberg group, $\phi(z, z) = |z|^2$, $Z_k = \partial_{z_k} - i\overline{z_k}\partial_t$, and $\Box_b$ is a diagonal operator (since $[Z_k, \overline{Z}_l] = 0$ when $k \neq l$). The above formula for $\Box_{LL}$ then gives the coefficient of $\Box_b$ acting on forms of the type $\phi_L(z)d\overline{z}^L$.

2.4 The $\Box_b$-Heat Equation and the Fourier Transform

The heat equation defined on $(0, q)$-forms on M is the initial value problem on $s \in (0, \infty)$ and $(z, t) \in M$ given by

$$\begin{cases} \frac{\partial\rho}{\partial s} + \Box_b\rho = 0 & \text{in } (0, \infty)\times M, \\ \rho(s = 0, z, t) = \delta_0(z, t) & \text{on } \{s = 0\}\times M. \end{cases}$$

Here, s is the time variable and $t \in \mathbb{R}^m$ is a spatial variable. Although we cannot find a closed form for $\rho(s, z, t)$, we can find the partial Fourier transform of $\rho(s, z, t)$ in the t-variables.

Given a variable $\tilde{t} \in \mathbb{R}$, the *(partial) Fourier transform* in $\tilde{t}$ is given by

$$\hat{f}(\tau) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\tilde{t}\tau} f(\tilde{t})\, d\tilde{t}.$$

If f is a function of several variables $f(\tilde{t}_1, \dots, \tilde{t}_k)$ and, for example, we take the partial Fourier transform in t_1, we use the notation $f(\widehat{\tau}, \tilde{t}_2, \dots, \tilde{t}_k)$.

As we will see below, to compute the partial Fourier transform of $\rho(s, z, t) = \rho_s(z, t)$, it is enough to solve the (Fourier transform of the) $\Box_{LL}$-heat equation

$$\begin{cases} \frac{\partial\rho}{\partial s} + \Box_{LL}\rho = 0 & \text{in } (0, \infty)\times M, \\ \rho_{s=0}(z, t) = \delta_0(z, t) & \text{on } \{s = 0\}\times M. \end{cases} \tag{3}$$

We start by computing the partial Fourier transform in t of $\Box_{LL}$, denoted $\Box^{\lambda}_{LL}$. We start with a reexamination of (2). By taking the partial Fourier transform in t of the formula for $\Box_{LL}$, the effect is to replace D_t with $i\lambda$. If we write $z = \sum_{k=1}^{n} z_k v_k$, then

$$\mathrm{Im}\left\{\sum_{k=1}^{n}\phi(z, v_k)\partial_{z_k}\right\}\cdot i\lambda = \sum_{k=1}^{n}\phi(v_k, v_k)\cdot i\lambda\,\mathrm{Im}\{z_k\partial_{z_k}\} = \sum_{k=1}^{n} i\mu_k^{\lambda}\,\mathrm{Im}\{z_k\partial_{z_k}\}$$

and

$$\sum_{k=1}^{n}\big(\phi(z,v_k)\cdot i\lambda\big)\big(\overline{\phi(z,v_k)}\cdot i\lambda\big)=\sum_{k=1}^{n}\big(z_k\phi(v_k,v_k)\cdot i\lambda\big)\big(\bar{z}_k\phi(v_k,v_k)\cdot i\lambda\big)$$
$$=-\sum_{k=1}^{n}(\mu_k^\lambda)^2|z_k|^2.$$

Consequently, (2) transforms to

$$\Box_{LL}^\lambda=-\frac{1}{4}\Delta+2i\sum_{k=1}^{n}\mu_k^\lambda\,\mathrm{Im}\{z_k\partial_{z_k}\}+\sum_{k=1}^{n}(\mu_k^\lambda)^2|z_k|^2-\Big(\sum_{k\in L}\mu_k^\lambda-\sum_{k\notin L}\mu_k^\lambda\Big). \quad (4)$$

We employ the following notation: for $1\le j\le\nu(\lambda)$, define $\epsilon_j^\lambda(L)=\epsilon_j^\lambda=\mathrm{sgn}(\mu_j^\lambda)$, if $j\in L$ and $\epsilon_j^\lambda=-\mathrm{sgn}(\mu_j^\lambda)$ if $j\notin L$.

Our main result is the following.

Theorem 1 *For any $\lambda\in\mathbb{R}^m$, the partial Fourier transform of the fundamental solution to the $\Box_{LL}$-heat equation satisfies the heat equation*

$$\begin{cases}\frac{\partial\rho}{\partial s}+\Box_{LL}^\lambda\rho=0 & \textit{in } (0,\infty)\times\mathbb{C}^n,\\ \rho(s=0,z,\widehat{\lambda})=(2\pi)^{-m/2}\delta_0(z) & \textit{on } \{s=0\}\times\mathbb{C}^n\end{cases}$$

and is given by

$$\rho(s,x,y,\widehat{\lambda})=\frac{2^{n-\nu(\lambda)}(2\pi)^{-(m/2+n)}}{s^{n-\nu(\lambda)}}e^{-\frac{|x''|^2+|y''|^2}{s}}$$
$$\times\prod_{j=1}^{\nu(\lambda)}\frac{2e^{s\epsilon_j^\lambda|\mu_j^\lambda|}\mu_j^\lambda}{\sinh(s\mu_j^\lambda)}e^{-\mu_j^\lambda\coth(\mu_j^\lambda s)(x_j^2+y_j^2)}.$$

Note that μ_j^λ and $\coth(s\mu_j^\lambda)$ are real-valued and are odd in μ_j^λ, so putting absolute values around the μ_j^λ would not change the result. Therefore, there is Gaussian decay in $(x_j^2+y_j^2)$ for all j when $\lambda\in\mathbb{R}^m$. Theorem 1 generalizes the case of Theorem 1.2 in [4] where (in the notation given there) $\tau\in\mathbb{R}$ and $\gamma=n-2q$.

We now cast the heat equation in terms of a weighted $\bar\partial$-problem in $\mathbb{C}^n$. Recall that $\overline{Z}_j=\frac{\partial}{\partial\bar{z}_j}+i\phi(z,v)\cdot D_t$. If we denote a superscript λ for the partial Fourier transform in t, then

$$\overline{Z}_j\mapsto\overline{Z}_j^\lambda=\frac{\partial}{\partial\bar{z}_j}-\phi(z,v_j)\cdot\lambda=e^{\phi(z,z)\cdot\lambda}\frac{\partial}{\partial\bar{z}_j}e^{-\phi(z,z)\cdot\lambda}.$$

From the computation of $\overline{Z}_j$, the tangential Cauchy-Riemann operator $\overline{\partial}_b$ is defined on $(0, q)$-forms on G by

$$\overline{\partial}_b f(z) = \sum_{\substack{K \in \mathcal{I}_{q+1} \\ J \in \mathcal{I}_q}} \sum_{j=1}^{n} \epsilon_K^{jJ} \overline{Z}_j f_J(z)\, d\overline{z}^K$$

where

$$\epsilon_K^{jJ} = \begin{cases} (-1)^\sigma & \text{if } \{j\} \cup J = K \text{ as sets and } \sigma \text{ is the sign of} \\ & \text{the permutation taking } \{j\} \cup J \text{ to } K, \\ 0 & \text{otherwise.} \end{cases}$$

This means that if g is a $(0, q)$-form in $\mathbb{C}^n$ and we treat λ as a parameter, then the partial Fourier transform in t of $\overline{\partial}_b$, denoted by $\overline{\partial}_b^\lambda$ is given by

$$\overline{\partial}_b^\lambda g(z) = e^{\phi(z,z)\cdot\lambda} \overline{\partial}\{e^{-\phi(z,z)\cdot\lambda} g\}$$

where $\overline{\partial}$ is the usual Cauchy-Riemann operator on $\mathbb{C}^n$. Since $\Box_b = \overline{\partial}_b \overline{\partial}_b^* + \overline{\partial}_b^* \overline{\partial}_b$ where $\overline{\partial}_b^*$ is the L^2-adjoint of $\overline{\partial}_b$, it follows that $\Box_b^\lambda = \overline{\partial}_b^\lambda (\overline{\partial}_b^\lambda)^* + (\overline{\partial}_b^\lambda)^* \overline{\partial}_b^\lambda$. Thus, solving for the $\Box_b^\lambda$-heat kernel also yields the heat kernel associated with the weighted $\overline{\partial}$-problem on $\mathbb{C}^n$ with the weight $e^{-\phi(z,z)\cdot\lambda}$.

Corollary 1 *For any* $\lambda \in \mathbb{R}^m$, *the function*

$$H^\lambda(s, z, \tilde{z}) = (2\pi)^{m/2} \rho_s(z - \tilde{z}, \widehat{\lambda}) e^{-2i\lambda\cdot \mathrm{Im}\,\phi(z,\tilde{z})}$$

satisfies the following: *if*

$$H^\lambda\{f\}(s, z) = \int_{\mathbb{C}^n} H^\lambda(s, z, \tilde{z}) f(\tilde{z})\, d\tilde{z},$$

then $H^\lambda\{f\}$ *solves the initial value problem for the weighted heat equation*:

$$\begin{cases} (\partial_s + \Box_b^\lambda) H^\lambda\{f\} = 0 & \text{in } (0, \infty) \times \mathbb{C}^n, \\ H^\lambda\{f\}(s = 0, z) = f(z) & \text{on } \{s = 0\} \times \mathbb{C}^n. \end{cases}$$

In particular, the component of $H^\lambda(s, z, \tilde{z})$ *on* $d\overline{z}^L$ *for* $L \in \mathcal{I}_q$ *is*

$$H_L^\lambda(s, z, \tilde{z}) = \frac{2^{n-\nu(\lambda)} (2\pi)^{-n}}{s^{n-\nu(\lambda)}} e^{-\frac{|z''-\tilde{z}''|^2}{s}} \times \prod_{j=1}^{\nu(\lambda)} \frac{2e^{s\epsilon_j^\lambda |\mu_j^\lambda|} \mu_j^\lambda}{\sinh(s\mu_j^\lambda)} e^{-\mu_j^\lambda \coth(\mu_j^\lambda s)|z_j - \tilde{z}_j|^2} e^{-2i\lambda\cdot \mathrm{Im}\,\phi(z,\tilde{z})}.$$

Note that the formula for the heat kernel yields a standard Gaussian solution for the Euclidean heat kernel in the zero eigenvalue directions. Also, the disappearance of the $(2\pi)^{-m/2}$ owes to the fact that $\delta_0(z, \widehat{\lambda}) = (2\pi)^{-m/2} \delta_0(z)$.

3 Representation Theory

3.1 Irreducible Unitary Representations

For $z = x + iy \in \mathbb{C}^n$, $t, \lambda \in \mathbb{R}^m$, and $\eta \in \mathbb{C}^{n-\nu(\lambda)}$, define $\pi_{\lambda,\eta}(x, y, t) : L^2(\mathbb{R}^{\nu(\lambda)}) \mapsto L^2(\mathbb{R}^{\nu(\lambda)})$ by

$$\pi_{\lambda,\eta}(x, y, t)(h)(\xi) = e^{i(\lambda\cdot t+2\,\mathrm{Re}(z''\cdot\overline{\eta}))} e^{-2i\sum_{j=1}^{\nu(\lambda)} \mu_j^\lambda y_j^\lambda(\xi_j + x_j^\lambda)} h(\xi + 2x')$$

for $h \in L^2(\mathbb{R}^{\nu(\lambda)})$ (so $\xi \in \mathbb{R}^{\nu(\lambda)}$). Note that if $\eta = \zeta + i\varsigma$, then $\mathrm{Re}(z'' \cdot \overline{\eta}) = x'' \cdot \zeta + y'' \cdot \varsigma$.

The map $\pi_{\lambda,\eta}(x, y, t)$ is unitary on $L^2(\mathbb{R}^{\nu(\lambda)})$. Also, π is a *representation* for G, which means that for each $\lambda \in \Omega$, $\pi_{\lambda,\eta}$ is a group homomorphism from G to the group of unitary operators on $L^2(\mathbb{R}^{\nu(\lambda)})$. Verifying that $\pi_{\lambda,\eta}$ is a representation and that all irreducible representations (up to equivalence) are of the form $\pi_{\lambda,\eta}$ is done in [13]. The formula for $\pi_{\lambda,\eta}$ is motivated by the Stone–von Neumann Theorem and its corollaries. On the Heisenberg group, it is explicitly worked out in [7].

If X is a right invariant vector field, then X gets "transformed" via $\pi_{\lambda,\eta}$ to an operator on $L^2(\mathbb{R}^{\nu(\lambda)})$ denoted by $T = d\pi_{\lambda,\eta}(X)$. This means that

$$X\{\pi_{\lambda,\eta}(g)\} = T \circ \pi_{\lambda,\eta}(g) \tag{5}$$

as operators on $L^2(\mathbb{R}^{\nu(\lambda)})$. It is usually easy to identify T by seeing what happens at $g = 0$ and using the right invariance of X to show that the above equation holds for all $g \in G$. To clarify, let $R_g(g') = g'g$ and recall that the vector field X at the point g is given by $X(g) = (R_g)_*\{X(0)\}$. If $X\{\pi_{\lambda,\eta}\}(0) = T \circ \pi_{\lambda,\eta}(0)$, then we have

$$\begin{aligned}
X\{\pi_{\lambda,\eta}\}(g) &= (R_g)_*\{X(0)\}\{\pi_{\lambda,\eta}(g)\} \\
&= X(g' = 0)\{\pi_{\lambda,\eta}(R_g(g'))\} \\
&= X(g' = 0)\{\pi_{\lambda,\eta}(g')\pi_{\lambda,\eta}(g)\} \quad \text{since } \pi \text{ is a homomorphism} \\
&= \{X(g' = 0)\pi_{\lambda,\eta}(g')\} \circ \pi_{\lambda,\eta}(g) \\
&= T \circ \pi_{\lambda,\eta}(g)
\end{aligned}$$

where the last equation uses the relationship of $X(g)$ and π at $g = 0$.

A similar computation shows that if X^ℓ is left invariant, then

$$X^\ell\{\pi_{\lambda,\eta}\}(g) = \pi_{\lambda,\eta}(g) \circ T$$

as operators on $L^2(\mathbb{R}^{\nu(\lambda)})$. Note that the order of T and $\pi_{\lambda,\eta}$ is reversed from (5). We will not dwell on this point as we prefer the use of right invariant vector fields. The relationship $X\{\pi_{\lambda,\eta}\}(g) = T \circ \pi_{\lambda,\eta}(g)$ is often expressed using the shorthand: $d\pi_{\lambda,\eta}(X) = T$.

From earlier, we have the right invariant vector fields

$$X_j = \partial_{v_j^\lambda} - 2\,\mathrm{Im}\,\phi(z, v_j^\lambda) \cdot D_t$$

and

$$Y_j = \partial_{Jv_j^\lambda} - 2\,\mathrm{Im}\,\phi(z, iv_j^\lambda)\cdot D_t = \partial_{Jv_j^\lambda} + 2\,\mathrm{Re}\,\phi(z, v_j^\lambda)\cdot D_t.$$

where J is the usual complex structure map on $\mathbb{R}^{2n} = \mathbb{C}^n$. In view of (5), we have the following relations: for

$$X_j\{\pi_{\lambda,\eta}\}(g) = \begin{cases} 2\partial_{\xi_j}\circ\pi_{\lambda,\eta}(g), & 1\le j\le \nu(\lambda),\\ 2i\zeta_j\circ\pi_{\lambda,\eta}(g), & \nu(\lambda)+1\le j\le n,\end{cases} \tag{6}$$

$$Y_j\{\pi_{\lambda,\eta}\}(g) = \begin{cases} -2i\mu_j^\lambda\xi_j\circ\pi_{\lambda,\eta}(g), & 1\le j\le \nu(\lambda),\\ 2i\varsigma_j\circ\pi_{\lambda,\eta}(g), & \nu(\lambda)+1\le j\le n,\end{cases} \tag{7}$$

$$\partial_{t_k}\{\pi_{\lambda,\eta}\}(g) = i\lambda_k\circ\pi_{\lambda,\eta}(g), \quad 1\le k\le m \tag{8}$$

as operators on $L^2(\mathbb{R}^{\nu(\lambda)})$. In the second equation, ξ_j is thought of as a multiplication operator on $L^2(\mathbb{R}^{\nu(\lambda)})$, i.e., $f(\xi)\mapsto f(\xi)\xi_j$. Equations (6) and (7) are easily shown to hold at the origin since $X_j(0) = \partial_{x_j}$ and $Y_j(0) = \partial_{y_j}$, and the right invariance forces these equations to hold at all $g\in G$.

Now we compute the "transform" of $\Box_{LK}$ in the coordinates $(z_1^\lambda,\dots,z_n^\lambda)$. Note that

$$d\pi_{\lambda,\eta}[Z_j,\overline{Z}_\ell] = \begin{cases} -2\mu_j^\lambda & \text{if } j=\ell,\\ 0 & \text{if } j\ne\ell.\end{cases}$$

This follows from (8) and the fact that the coordinates $(z_1^\lambda,\dots,z_n^\lambda)$ were chosen to diagonalize the form $\phi(z,\tilde z)\cdot\lambda$. In view of (1) and (6)–(8), we have

$$d\pi_{\lambda,\eta}\Box_{LK} = \begin{cases} -\Delta_\xi + |\eta|^2 + \sum_{j=1}^{\nu(\lambda)}(\mu_j^\lambda)^2\xi_j^2 - \sum_{j=1}^{\nu(\lambda)}\epsilon_j^\lambda|\mu_j^\lambda| & \text{if } K=L,\\ 0 & \text{if } K\ne L.\end{cases} \tag{9}$$

We will also need to transform the adjoint of $\Box_{LK}$ which is defined as

$$\int_{(z,t)\in G}\Box_{LK}\{f(z,t)\}g(z,t)\,dx\,dy\,dt = \int_{(z,t)\in G} f(z,t)\Box_{LK}^{\mathrm{adj}}\{g(z,t)\}\,dx\,dy\,dt$$

(note: this is the "integration by parts" adjoint, not the L^2 adjoint, since there is no conjugation). We have

$$\begin{aligned} Q_\xi^{\lambda,\eta,LK} &:= d\pi_{\lambda,\eta}\Box_{LK}^{\mathrm{adj}}\\ &= \begin{cases} -\Delta_\xi + |\eta|^2 + \sum_{j=1}^{\nu(\lambda)}(\mu_j^\lambda)^2\xi_j^2 + \sum_{j=1}^{\nu(\lambda)}\epsilon_j^\lambda|\mu_j^\lambda| & \text{if } K=L,\\ 0 & \text{if } K\ne L\end{cases}\end{aligned} \tag{10}$$

(just a sign change for the last term on the right). The subscript ξ on $Q_\xi^{\lambda,\eta,LK}$ indicates that this is a differential operator in the ξ variable (instead of the group variable $g=(x,y,t)$). Below, we assume $L=K$ (otherwise the operator is zero) and that L,

λ, and η are fixed. We drop the superscript LL when its use is unambiguous. In view of (1) and (6)–(8), we have

$$\Box_{LL}^{\text{adj}}\{\pi_{\lambda,\eta}(g)\} = Q_\xi^{\lambda,\eta} \circ \pi_{\lambda,\eta}(g) \tag{11}$$

as operators on $L^2(\mathbb{R}^{\nu(\lambda)})$. We return to this key equation later.

3.2 Group Fourier Transform

For $(z,t) \in G$, we express $(z,t) = (x,y,t) = (x',y',x'',y'',t) = (x',y',z'',t)$. The variable z'' may be thought of as in $\mathbb{C}^{n-\nu(\lambda)}$ or $\mathbb{R}^{2(n-\nu(\lambda))}$.

For $f : G \mapsto \mathbb{C}$, we define the *group Fourier transform* of f as the operator $T_f^{\lambda,\eta}$: $L^2(\mathbb{R}^{\nu(\lambda)}) \mapsto L^2(\mathbb{R}^{\nu(\lambda)})$ where for $h \in L^2(\mathbb{R}^{\nu(\lambda)})$,

$$\begin{aligned} T_f^{\lambda,\eta}\{h\}(\xi) &= \int_{(z=x+iy,t)\in G} f(z,t)\pi_{\lambda,\eta}(z,t)(h)(\xi)\,dx\,dy\,dt \\ &= \int_{(z=x+iy,t)\in G} f(z,t)e^{i(\lambda\cdot t+2\,\mathrm{Re}(z''\cdot\overline{\eta}))}e^{-2i\sum_{j=1}^{\nu(\lambda)}\mu_j^\lambda y_j^\lambda(\xi_j+x_j^\lambda)} \\ &\quad \times h(\xi+2x')\,dx\,dy\,dt. \end{aligned}$$

As before, x_j, y_j are the coordinates for $x, y \in \mathbb{R}^n$ relative to the basis $v_1^\lambda, \ldots, v_n^\lambda$. Note that

$$\begin{aligned} T_f^{\lambda,\eta}\{h\}(\xi) &= (2\pi)^{(2n+m-\nu(\lambda))/2} \\ &\quad \times \int_{x'\in\mathbb{R}^{\nu(\lambda)}} f(x', \widehat{2\mu^\lambda\circ(\xi+x')}, \widehat{-2\eta}, \widehat{-\lambda})h(\xi+2x')\,dx'. \end{aligned}$$

We have written $\mu^\lambda \circ (\xi + x')$ for $(\mu_1^\lambda(\xi_1 + x_1^\lambda), \ldots, \mu_{\nu(\lambda)}^\lambda(\xi_{\nu(\lambda)} + x_{\nu(\lambda)}^\lambda))$. We can also express $T_f^{\lambda,\eta}\{h\}$ as

$$\begin{aligned} &T_f^{\lambda,\eta}\{h\}(\xi) \\ &= (2\pi)^{(2n+m-\nu(\lambda))/2} \\ &\quad \times \int_{x'\in\mathbb{R}^{\nu(\lambda)}} \mathcal{F}_{x'',y,t}\{f(x,y,t)e^{-2i\sum_{j=1}^{\nu(\lambda)}\mu_j^\lambda x_j y_j}\}(x', 2\mu^\lambda\circ\xi, -2\eta, -\lambda)h(\xi+2x')\,dx'. \end{aligned} \tag{12}$$

In the above notation, $\mathcal{F}_{x'',y,t}$ indicates the Fourier transform in the (x'', y, t) variables only, whereas $\mathcal{F}$ indicates the Fourier transform in all variables (except s).

In view of (11), we have

$$\begin{aligned} Q_\xi^{\lambda,\eta}\{T_f^{\lambda,\eta}(h)(\xi)\} &= \int_{(z=x+iy,t)\in G} f(z,t)\Box_{LL}^{\text{adj}}\{\pi_{\lambda,\eta}(z,t)h(\xi)\}\,dx\,dy\,dt \\ &= \int_{(z=x+iy,t)\in G} \Box_{LL}\{f(z,t)\}\pi_{\lambda,\eta}(z,t)h(\xi)\,dx\,dy\,dt. \end{aligned} \tag{13}$$

4 The Heat Equation

4.1 The Heat Equation on M

Our goal is to find a formula for the fundamental solution to the heat equation (3). We know abstractly that ρ exists: $\Box_{LL}$ is self-adjoint and nonnegative, so $e^{-s\Box_{LL}}$ is a well-defined, bounded linear operator on $L^2(G)$ with norm at most 1. It has an integral kernel by the Schwartz kernel theorem. For the computations performed here, it suffices to assume that the ρ is smooth and in L^2 because an *a posteriori* computation verifies that ρ is the unique fundamental solution to the $\Box_{LL}$-heat equation.

Let us apply the group Fourier transform to ρ and recall that $\rho_s(z,t) = \rho(s,z,t)$. Define the operator $U^{\lambda,\eta}(s) : L^2(\mathbb{R}^{\nu(\lambda)}) \mapsto L^2(\mathbb{R}^{\nu(\lambda)})$ by

$$U^{\lambda,\eta}(s)\{h\}(\xi) = T^{\lambda,\eta}_{\rho_s}\{h\}(\xi) = \int_{(z,t)\in G} \rho_s(z,t)\pi_{\lambda,\eta}(z,t)h(\xi)\,dz\,dt. \tag{14}$$

In view of (13) and the fact that $\rho_s(z,t)$ solves the heat equation, we have

$$\begin{aligned} Q^{\lambda,\eta}_{\xi}\{U^{\lambda,\eta}(s)\{h\}(\xi)\} &= \int_{(z,t)\in G} \Box_{LL}\{\rho_s(z,t)\}\pi_{\lambda,\eta}(z,t)h(\xi)\,dz\,dt \\ &= -\partial_s\left\{\int_{(z,t)\in G} \rho_s(z,t)\pi_{\lambda,\eta}(z,t)h(\xi)\,dz\,dt\right\} \\ &= -\partial_s\left\{U^{\lambda,\eta}(s)\{h\}(\xi)\right\} \end{aligned}$$

Also

$$U^{\lambda,\eta}(s=0)\{h\}(\xi) = T^{\lambda,\eta}_{\delta_0}\{h\}(\xi) = h(\xi).$$

Therefore, we conclude that $U^{\lambda,\eta}(s)$ satisfies the following boundary value problem:

$$Q^{\lambda,\eta}_{\xi}\{U^{\lambda}(s)\} = -\partial_s\{U^{\lambda}(s)\} \quad \text{and} \quad U^{\lambda}(s=0) = \mathrm{Id} \tag{15}$$

where Id is the identity operator on $L^2(\mathbb{R}^{\nu(\lambda)})$. This is a Hermite equation similar to, though more complicated than, the one we solved in the Heisenberg group case [4]. So, our approach is to proceed as follows: **1)** explicitly solve this Hermite equation, and then **2)** recover the fundamental solution to the heat equation.

As to the second task, we let $a \in \mathbb{R}^{\nu(\lambda)}$ be an arbitrary vector, and then define $h_a(\xi) = (2\pi)^{-n-m/2}e^{-i\xi\cdot a}$. Let

$$u^{\lambda,\eta}(s,a,\xi) = U^{\lambda,\eta}(s)\{h_a\}(\xi).$$

The above definition needs explanation since $h_a \notin L^2(\mathbb{R}^{\nu(\lambda)})$. For each fixed $s > 0$, $\rho_s \in L^2(G)$ and we can approximate ρ_s by $\rho^{\delta}_s \in L^1 \cap L^2(G)$ (e.g., by multiplying ρ_s with an appropriate test function). Then, as we see below, we can define $U^{\lambda,\eta}_{\delta}(s)\{h_a\}(\xi) = T^{\lambda,\eta}_{\rho^{\delta}_s}\{h_a\}(\xi)$ since in view of (12),

$$U_\delta^{\lambda,\eta}(s)\{h_a\}(\xi)$$
$$= \frac{1}{(2\pi)^{\nu(\lambda)/2}}$$
$$\times \int_{x'\in\mathbb{R}^{\nu(\lambda)}} \mathcal{F}_{x'',y,t}\big\{\rho_s^\delta(x,y,t)e^{-2i\sum_{j=1}^{\nu(\lambda)}\mu_j^\lambda y_j x_j}\big\}(x',2\mu^\lambda\circ\xi,-2\eta,-\lambda)e^{-i(\xi+2x')\cdot a}\,dx'$$
$$= \mathcal{F}\big\{\rho_s^\delta(x,y,t)e^{-2i\sum_{j=1}^{\nu(\lambda)}\mu_j^\lambda y_j x_j}\big\}(2a,2\mu^\lambda\circ\xi,-2\eta,-\lambda)e^{-i\xi\cdot a}.$$

By the definition of the Fourier transform in L^2,

$$\mathcal{F}\big\{\rho_s^\delta(x,y,t)e^{-2i\sum_{j=1}^{\nu(\lambda)}\mu_j^\lambda y_j x_j}\big\}(2a,2\mu^\lambda\circ\xi,-2\eta,-\lambda)e^{-i\xi\cdot a}$$
$$\longrightarrow \mathcal{F}\big\{\rho_s(x,y,t)e^{-2i\sum_{j=1}^{\nu(\lambda)}\mu_j^\lambda y_j x_j}\big\}(2a,2\mu^\lambda\circ\xi,-2\eta,-\lambda)e^{-i\xi\cdot a}$$

in $L^2(\mathbb{R}^{\nu(\lambda)})$ as $\delta\to 0$. Thus, $u^{\lambda,\eta}(s,a,\xi)$ is well-defined. In the above computation, we view $\eta=(\zeta,\varsigma)\in\mathbb{R}^{2(n-\nu(\lambda))}$. Also, the motivation for the choice of $h=h_a$ is that it offers the "missing" exponential needed to relate the full Fourier transform of ρ_s with $u^{\lambda,\eta}$. Now it is just a matter of unraveling the equation

$$u^{\lambda,\eta}(s,a,\xi)=\mathcal{F}\big\{\rho_s(x,y,t)e^{-2i\sum_{j=1}^{\nu(\lambda)}\mu_j^\lambda y_j^\lambda x_j^\lambda}\big\}(2a,2\mu^\lambda\circ\xi,-2\eta,-\lambda)e^{-i\xi\cdot a} \tag{16}$$

for ρ_s using the inverse Fourier transform.

Before we go on, let us remark that had we used left invariant vector fields rather than right invariant ones, then the transformed operator, $Q_\xi^{\lambda,\eta}$ would appear on the right of the group transform. That is to say, we would be trying to solve the following analogue of (15)

$$\partial_s\{T_{\rho_s}^\lambda\}=-T_{\rho_s}^\lambda\,\tilde{Q}_\xi^{\lambda,\eta}\quad\text{and}\quad T_{\rho_{s=0}}^\lambda=\mathrm{Id}$$

where $\tilde{Q}_\xi^{\lambda,\eta}$ is a Hermite type differential operator similar to $Q_\xi^{\lambda,\eta}$. Note the transform operator T^λ is now intertwined with the differential operators (i.e., ∂_s is on the left side and $\tilde{Q}_\xi^{\lambda,\eta}$ is on the right). Since the inversion formula for the group transform operator is complicated (see [13]), it would appear that using left invariant vector fields makes it more difficult to unravel a formula for ρ.

4.2 Weighted Heat Equation

Our objective is to compute $\rho_s(x',y',\widehat{\eta},\widehat{\lambda})$ by solving the weighted heat equation obtained by taking the partial Fourier transform in the t and (x'',y'')-variables. We obtain the $\Box_{LL}^{\lambda,\eta}$-heat equation

$$\begin{cases}\partial_s\rho_s(x',y',\widehat{\eta},\widehat{\lambda})=-\Box_{LL}^{\lambda,\eta}\rho_s(x',y',\widehat{\eta},\widehat{\lambda}),\\ \rho_{s=0}(x',y',\widehat{\eta},\widehat{\lambda})=(2\pi)^{-m/2-(n-\nu(\lambda))}\delta_0(x',y').\end{cases}$$

From (4), we have

$$\Box_{LL}^{\lambda,\eta} = -\frac{1}{4}\Delta + \frac{1}{4}|\eta|^2 + 2i\sum_{j=1}^{\nu(\lambda)} \mu_j^\lambda \operatorname{Im}\{z_j \partial_{z_j}\} + \sum_{j=1}^{\nu(\lambda)} |z_j \mu_j^\lambda|^2 - \sum_{j=1}^{\nu(\lambda)} \epsilon_j^\lambda |\mu_j^\lambda|.$$

In the following computation, we find a formula of $\rho_s(x', y', \widehat{\eta}, \widehat{\lambda})$. Observe that $\mu_j^{-\lambda} = -\mu_j^\lambda$ and $\epsilon_j^{-\lambda} = -\epsilon_j^\lambda$. (Note that $v_j^{-\lambda} = v_j^\lambda$, so we can continue to suppress the λ superscript on x_j and y_j.) We unravel (16) to obtain (with $a, b \in \mathbb{R}^{\nu(\lambda)}$)

$$\rho_s(x', y', \widehat{\eta}, \widehat{\lambda}) = e^{-2i\sum_{j=1}^{\nu(\lambda)} \mu_j^\lambda x_j y_j} \mathcal{F}_{a,b}^{-1}\big(e^{-\frac{i}{4}\sum_{j=1}^{\nu(\lambda)} a_j b_j/\mu_j^\lambda} \tilde{u}^{\lambda,\eta}(s, a, b)\big)(x', y') \quad (17)$$

where $\tilde{u}^{\lambda,\eta}(s, a, b) = u^{-\lambda, -\frac{1}{2}\eta}(s, a/2, b/(2\mu^{-\lambda}))$ and $b/(2\mu^{-\lambda})$ is the vector quantity whose jth component is $b_j/(2\mu_j^{-\lambda})$. As we shall see, the inverse Fourier transform in the a and b variables will be relatively simple (using Gaussian integrals). In the next section, we use Hermite functions to solve for $\tilde{u}^{\lambda,\eta}$ on the "transform" side. Then we return to the above formula to compute $\rho_s(x', y', \widehat{\eta}, \widehat{\lambda})$.

5 Computing the Heat Kernels

In this section, we prove Theorem 1 and Corollary 1.

5.1 Hermite Functions

Our starting point is (15), which we restate as $Q_\xi^{\lambda,\eta}\{U^{\lambda,\eta}(s)\} = -\partial_s\{U^{\lambda,\eta}(s)\}$ where

$$Q_\xi^{\lambda,\eta} = -\Delta_\xi + |\eta|^2 + \sum_{j=1}^{\nu(\lambda)} (\mu_j^\lambda \xi_j)^2 + \sum_{j=1}^{\nu(\lambda)} \epsilon_j^\lambda |\mu_j^\lambda|.$$

We use Hermite functions to solve this equation. For a nonnegative integer ℓ, define

$$\psi_\ell(x) = \frac{(-1)^\ell}{2^{\ell/2}\pi^{1/4}(\ell!)^{1/2}} \frac{d^\ell}{dx^\ell}\{e^{-x^2}\}e^{x^2/2}, \quad x \in \mathbb{R}.$$

Each ψ_ℓ has unit L^2-norm on the real line and satisfies the equation

$$-\psi_\ell''(x) + x^2\psi_\ell(x) = (2\ell+1)\psi_\ell(x);$$

see [19], (1.1.9). For $\lambda \in \mathbb{R}^m \setminus \{0\}$, define

$$\psi_{\ell_j}^\lambda(\xi_j) = \psi_{\ell_j}(|\mu_j^\lambda|^{1/2}\xi_j)|\mu_j^\lambda|^{1/4}.$$

Each $\psi_{\ell_j}^\lambda(\xi_j)$ has unit L^2-norm on $\mathbb{R}$ and hence ψ_ℓ^λ has unit L^2-norm on $\mathbb{R}^{\nu(\lambda)}$. An easy calculation shows that

$$(-\partial_{\xi_j\xi_j} + (\mu_j^\lambda \xi_j)^2)\{\psi_{\ell_j}^\lambda(\xi_j)\} = (2\ell_j + 1)\psi_{\ell_j}^\lambda(\xi_j)|\mu_j^\lambda|. \quad (18)$$

For $s > 0$, we claim that $U^{\lambda,\eta}(s) : L^2(\mathbb{R}^{\nu(\lambda)}) \mapsto L^2(\mathbb{R}^{\nu(\lambda)})$ as defined in (14) is given by

$$U^{\lambda,\eta}(s) = e^{-s|\eta|^2} \bigotimes_{j=1}^{\nu(\lambda)} \sum_{\ell_j=0}^{\infty} e^{-[(2\ell_j+1)+\epsilon_j^\lambda]|\mu_j^\lambda|s} P_{\ell_j}^\lambda$$

where $P_{\ell_j}^\lambda$ is the L^2 projection of a smooth function of polynomial growth in the variable ξ_j onto the space spanned by $\psi_{\ell_j}^\lambda(\xi_j)$, and where $\bigotimes_{j=1}^{\nu(\lambda)}$ is the tensor product (so that the output of $U^{\lambda,\eta}(s)$ is a function of $\xi_1, \dots, \xi_{\nu(\lambda)}$). For shorthand, we write

$$E_{\ell_j}^\lambda(s) = e^{-[(2\ell_j+1)+\epsilon_j^\lambda]|\mu_j^\lambda|s}.$$

We then have $U^{\lambda,\eta}(s) = e^{-s|\eta|^2} \bigotimes_{j=1}^{\nu(\lambda)} \sum_{\ell_j=0}^{\infty} E_{\ell_j}^\lambda(s) P_{\ell_j}^\lambda$. Using the product rule, we compute

$$\begin{aligned}
&\partial_s\{U^{\lambda,\eta}(s)\} \\
&\quad = -|\eta|^2 U^{\lambda,\eta}(s) + e^{-s|\eta|^2} \sum_{j=1}^{\nu(\lambda)} \partial_s \left(\sum_{\ell_j=0}^{\infty} E_{\ell_j}^\lambda(s) P_{\ell_j}^\lambda \right) \bigotimes_{\substack{k=1 \\ k\neq j}}^{\nu(\lambda)} \sum_{\ell_k=0}^{\infty} E_{\ell_k}^\lambda(s) P_{\ell_k}^\lambda \\
&\quad = -|\eta|^2 U^{\lambda,\eta}(s) + e^{-s|\eta|^2} \sum_{j=1}^{\nu(\lambda)} \sum_{\ell_j=0}^{\infty} -[(2\ell_j+1)+\epsilon_j^\lambda]|\mu_j^\lambda| \\
&\qquad \times e^{-[(2\ell_j+1)+\epsilon_j^\lambda]|\mu_j^\lambda|s} P_{\ell_j}^\lambda \bigotimes_{\substack{k=1 \\ k\neq j}}^{\nu(\lambda)} \sum_{\ell_k=0}^{\infty} E_{\ell_k}^\lambda(s) P_{\ell_k}^\lambda \\
&\quad = -|\eta|^2 U^{\lambda,\eta}(s) + e^{-s|\eta|^2} \sum_{j=1}^{\nu(\lambda)} \sum_{\ell_j=0}^{\infty} \left(\partial_{\xi_j\xi_j} - (\mu_j^\lambda \xi_j)^2 - \epsilon_j^\lambda |\mu_j^\lambda|\right) \\
&\qquad \circ e^{-[(2\ell_j+1)+\epsilon_j^\lambda]|\mu_j^\lambda|s} P_{\ell_j}^\lambda \bigotimes_{\substack{k=1 \\ k\neq j}}^{\nu(\lambda)} \sum_{\ell_k=0}^{\infty} E_{\ell_k}^\lambda(s) P_{\ell_k}^\lambda
\end{aligned}$$

where the last equality uses (18). Since the differential operator on the right is independent of ℓ_j, we can factor it to the left of $\sum_{\ell_j}$ to obtain

$$\partial_s\{U^{\lambda,\eta}(s)\} = -Q_\xi^{\lambda,\eta}\{U^{\lambda,\eta}(s)\}.$$

Since the Hermite functions, ψ_ℓ^λ, form an orthonormal basis for $L^2(\mathbb{R})$, $U^{\lambda,\eta}(s=0)$ is just the identity operator. Thus $U^{\lambda,\eta}(s)$ solves (15).

As above, we apply $U^{\lambda,\eta}(s)$ to the function $h_a(\xi) = (2\pi)^{-n-m/2} e^{-i\xi\cdot a}$ to obtain the fundamental solution ρ_s. We therefore obtain

$$u^{\lambda,\eta}(s,a,\xi) = U^{\lambda,\eta}(s)\{h_a(\xi)\}$$
$$= (2\pi)^{-n-m/2} e^{-s|\eta|^2} \prod_{j=1}^{\nu(\lambda)} \sum_{\ell_j=0}^{\infty} E^\lambda_{\ell_j}(s) P^\lambda_{\ell_j}\{e^{-i\xi_j a_j}\}.$$

Since h_a belongs to $L^\infty(\mathbb{R}^{\nu(\lambda)})$ and *not* in $L^2(\mathbb{R}^{\nu(\lambda)})$, the above sum converges *a priori* in the sense of tempered distributions (as opposed to L^2 convergence). Earlier, we argued that we can obtain $U^{\lambda,\eta}(s)\{h_a\}$ via a standard approximation argument, however, we will see below that the convergence is much stronger and the result is a smooth function in $s,\ a,\ \xi$. Each projection term on the right is

$$P^\lambda_{\ell_j}(e^{-i\xi_j a_j}) = \left(\int_{\tilde{\xi}_j\in\mathbb{R}} e^{-i\tilde{\xi}_j a_j} \psi_{\ell_j}(|\mu^\lambda_j|^{1/2}\tilde{\xi}_j)|\mu^\lambda_j|^{1/4}\, d\tilde{\xi}_j\right)|\mu^\lambda_j|^{1/4}\psi_{\ell_j}(|\mu^\lambda_j|^{1/2}\xi_j)$$
$$= (2\pi)^{1/2}\widehat{\psi_{\ell_j}}(a_j/|\mu^\lambda_j|^{1/2})\psi_{\ell_j}(|\mu^\lambda_j|^{1/2}\xi_j)$$
$$= (2\pi)^{1/2}(-i)^{\ell_j}\psi_{\ell_j}(a_j/|\mu^\lambda_j|^{1/2})\psi_{\ell_j}(|\mu^\lambda_j|^{1/2}\xi_j)$$

where the last equality uses a standard fact about Hermite functions that they equal their Fourier transforms up to a factor of $(-i)^{\ell_j}$. Substituting this expression on the right into the definition of $u^{\lambda,\eta}(s,a,\xi)$, we obtain

$$u^{\lambda,\eta}(s,a,\xi) = (2\pi)^{-n-m/2+\nu(\lambda)/2} e^{-s|\eta|^2}$$
$$\times \prod_{j=1}^{\nu(\lambda)} \sum_{\ell_j=0}^{\infty} E^\lambda_{\ell_j}(s)(-i)^{\ell_j}\psi_{\ell_j}(a_j/|\mu^\lambda_j|^{1/2})\psi_{\ell_j}(|\mu^\lambda_j|^{1/2}\xi_j).$$

This function satisfies

$$\partial_s u^{\lambda,\eta}(s,a,\xi) = -Q^{\lambda,\eta}_\xi\{u^{\lambda,\eta}(s,a,\xi)\},$$
$$u^{\lambda,\eta}(s=0,a,\xi) = h_a(\xi) = (2\pi)^{-n-m/2}e^{-ia\cdot\xi}.$$

In view of (17), for computing $\rho_s(x',y',\widehat{\eta},\widehat{\lambda})$, we need to compute

$$\tilde{u}^{\lambda,\eta}(s,a,b) = u^{-\lambda,-\frac{1}{2}\eta}(s,a/2,b/(2\mu^{-\lambda}))$$

where $b/(2\mu^{-\lambda})$ is the vector quantity whose jth component is $b_j/(2\mu^{-\lambda}_j)$. From the previous equality, and using that $\mu^{-\lambda}_j = -\mu^\lambda_j$, $\epsilon^{-\lambda}_j = -\epsilon^\lambda_j$, we have

$$\tilde{u}^{\lambda,\eta}(s,a,b) = (2\pi)^{-\frac{1}{2}(n+m+(n-\nu(\lambda)))} e^{-s\frac{|\eta|^2}{4}} \prod_{j=1}^{\nu(\lambda)} e^{-(1-\epsilon^\lambda_j)|\mu^\lambda_j|s}$$
$$\times \sum_{\ell_j=0}^{\infty} (-i)^{\ell_j}\psi_{\ell_j}(a_j/2|\mu^\lambda_j|^{1/2})\psi_{\ell_j}(b_j|\mu^\lambda_j|^{1/2}/2\mu^{-\lambda}_j)e^{-2\ell_j|\mu^\lambda_j|s}.$$

Let

$$S_j = e^{-2|\mu_j^\lambda|s}, \qquad \alpha_j = \frac{a_j}{2|\mu_j^\lambda|^{1/2}}, \qquad \beta_j = \frac{-b_j|\mu_j^\lambda|^{1/2}}{2\mu_j^\lambda}. \tag{19}$$

Then

$$\tilde{u}^{\lambda,\eta}(s,a,b) = (2\pi)^{-\frac{1}{2}(n+m+(n-\nu(\lambda)))} e^{-s\frac{|\eta|^2}{4}} \prod_{j=1}^{\nu(\lambda)} S_j^{(1-\epsilon_j^\lambda)/2} \sum_{\ell=0}^{\infty} (-iS_j)^\ell \psi_\ell(\alpha_j)\psi_\ell(\beta_j).$$

Using Mehler's formula ([19], Lemma 1.1.1) for Hermite functions, we obtain

$$\tilde{u}^{\lambda,\eta}(s,a,b) = (2\pi)^{-(m/2+n)} 2^{\nu(\lambda)/2} e^{-s\frac{|\eta|^2}{4}} \prod_{j=1}^{\nu(\lambda)} S_j^{(1-\epsilon_j^\lambda)/2}$$

$$\times \frac{1}{\sqrt{1+S_j^2}} e^{-\frac{1}{2}(\frac{1-S_j^2}{1+S_j^2})(\alpha_j^2+\beta_j^2) - \frac{2iS_j\alpha_j\beta_j}{1+S_j^2}}.$$

The series for $\tilde{u}^{\lambda,\eta}$ converges in C^∞ on the unit disk in $\mathbb{C}$, and therefore the series for $\tilde{u}^{\lambda,\eta}$ converges in $\mathbb{C}^\infty$ for $s > 0$, justifying many previous computations (which held *a priori* in the category of tempered distributions).

5.2 Finishing the Proof of Theorem 1

In view of (17), to determine $\rho_s(x', y', \widehat{\eta}, \widehat{\lambda})$, we must compute

$$\mathcal{F}^{-1}_{a,b}\big(e^{-i\sum_{j=1}^{\nu(\lambda)} a_j b_j/(4\mu_j^\lambda)} \tilde{u}^{\lambda,\eta}(s,a,b)\big)(x', y').$$

Using (19) and simplifying, we obtain

$$e^{-i\sum_{j=1}^{\nu(\lambda)} a_j b_j/(4\mu_j^\lambda)} \tilde{u}^{\lambda,\eta}(s,a,b)$$
$$= (2\pi)^{-(m/2+n)} e^{-s\frac{|\eta|^2}{4}} \prod_{j=1}^{\nu(\lambda)} \frac{e^{\epsilon_j^\lambda |\mu_j^\lambda| s}}{\sqrt{\cosh(2|\mu_j^\lambda|s)}} e^{-A_j(a_j^2+b_j^2)/2 - iB_j a_j b_j}$$

where

$$A_j = \frac{\tanh(2|\mu_j^\lambda|s)}{4|\mu_j^\lambda|}, \qquad B_j = \frac{\sinh^2(|\mu_j^\lambda|s)}{2\mu_j^\lambda \cosh(2|\mu_j^\lambda|s)}.$$

After an exercise in computing Gaussian integrals, we obtain

$$\mathcal{F}_{a,b}^{-1}\Big\{e^{-i\sum_{j=1}^{\nu(\lambda)} a_j b_j/(4\mu_j^\lambda)}\tilde{u}^{\lambda,\eta}(s,a,b)\Big\}(x',y')$$
$$= (2\pi)^{-(m/2+n)} e^{-s\frac{|\eta|^2}{4}} \prod_{j=1}^{\nu(\lambda)} \frac{e^{\epsilon_j^\lambda |\mu_j^\lambda| s}}{\sqrt{\cosh(2|\mu_j^\lambda|s)}} \frac{e^{\frac{-A_j}{2(A_j^2+B_j^2)}(x_j^2+y_j^2)+i\frac{B_j x_j y_j}{A_j^2+B_j^2}}}{\sqrt{A_j^2+B_j^2}}.$$

After simplifying,

$$\frac{-A_j}{2(A_j^2+B_j^2)} = -\mu_j^\lambda A_j/B_j, \qquad \frac{B_j}{A_j^2+B_j^2} = 2\mu_j^\lambda,$$
$$\sqrt{\cosh(2|\mu_j^\lambda|s)}\sqrt{A_j^2+B_j^2} = \frac{\sinh(s|\mu_j^\lambda|)}{2|\mu_j^\lambda|}.$$

The previous expression becomes

$$\mathcal{F}_{a,b}^{-1}\big(e^{-i\sum_{j=1}^{\nu(\lambda)} a_j b_j/(4\mu_j^\lambda)}\tilde{u}^{\lambda,\eta}(s,a,b)\big)(x',y')$$
$$= (2\pi)^{-(m/2+n)} e^{-s\frac{|\eta|^2}{4}} \prod_{j=1}^{\nu(\lambda)} \frac{2e^{\epsilon_j^\lambda|\mu_j^\lambda|s}|\mu_j^\lambda|}{\sinh(s|\mu_j^\lambda|)} e^{-\mu_j^\lambda(A_j/B_j)(x_j^2+y_j^2)+2i\mu_j^\lambda x_j y_j}.$$

In view of (17), the fundamental solution $\rho_s(x',y',\widehat{\eta},\widehat{\lambda})$ to the weighted heat equation is obtained by multiplying this expression by $\prod_{j=1}^{\nu(\lambda)} e^{-2i\mu_j^\lambda x_j y_j}$ which cancels the similar expression on the right side. We therefore obtain

$$\rho_s(x',y',\widehat{\eta},\widehat{\lambda}) = (2\pi)^{-(m/2+n)} e^{-s\frac{|\eta|^2}{4}} \prod_{j=1}^{\nu(\lambda)} \frac{2e^{\epsilon_j^\lambda|\mu_j^\lambda|s}|\mu_j^\lambda|}{\sinh(s|\mu_j^\lambda|)} e^{-\mu_j^\lambda(A_j/B_j)(x_j^2+y_j^2)}.$$

Note that the rightmost exponent can be rewritten as

$$-\mu_j^\lambda(A_j/B_j)(x_j^2+y_j^2) = -\frac{|\mu_j^\lambda|\sinh(2|\mu_j^\lambda|s)}{2\sinh^2(|\mu_j^\lambda|s)}(x_j^2+y_j^2) = -\mu_j^\lambda\coth(\mu_j^\lambda s)(x_j^2+y_j^2).$$

Consequently,

$$\rho_s(x,y,\widehat{\lambda}) = \frac{2^{n-\nu(\lambda)}(2\pi)^{-(m/2+n)}}{s^{n-\nu(\lambda)}} e^{-\frac{|x''|^2+|y''|^2}{s}}$$
$$\times \prod_{j=1}^{\nu(\lambda)} \frac{2e^{\epsilon_j^\lambda|\mu_j^\lambda|s}|\mu_j^\lambda|}{\sinh(s|\mu_j^\lambda|)} e^{-\mu_j^\lambda\coth(\mu_j^\lambda s)(x_j^2+y_j^2)}.$$

This completes the proof of Theorem 1 for $\lambda \in \Omega$.

5.3 The Proof of Corollary 1

In this subsection and the next, we show that the following kernel:

$$\begin{aligned} H^\lambda(s,z,\tilde{z}) &= (2\pi)^{m/2}\rho_s(z-\tilde{z},\widehat{\lambda})e^{-2i\lambda\cdot\mathrm{Im}\,\phi(z,\tilde{z})} \\ &= \frac{2^{n-\nu(\lambda)}(2\pi)^{-n}}{s^{n-\nu(\lambda)}} \\ &\quad \times e^{-\frac{|z''-\tilde{z}''|^2}{s}}\prod_{j=1}^{\nu(\lambda)}\frac{2e^{\epsilon_j^\lambda|\mu_j^\lambda|s}|\mu_j^\lambda|}{\sinh(s|\mu_j^\lambda|)}e^{-\mu_j^\lambda\coth(\mu_j^\lambda s)|z_j-\tilde{z}_j|^2}e^{-2i\lambda\cdot\mathrm{Im}\,\phi(z,\tilde{z})} \end{aligned} \tag{20}$$

is the heat kernel for the weighted $\overline{\partial}$-operator in $\mathbb{C}^n$. Here, $z = x + iy$ and $\tilde{z} = \tilde{x} + i\tilde{y}$. Note that H is conjugate symmetric, i.e., $H^\lambda(s,\tilde{z},z) = \overline{H^\lambda(s,z,\tilde{z})}$. We will show that the heat kernel has the following properties: if $f \in L^2(\mathbb{C}^n)$, then

$$H^\lambda\{f\}(s,x,y) = \int_{\mathbb{R}^n\times\mathbb{R}^n} H^\lambda(s,x,y,\tilde{x},\tilde{y})f(\tilde{x},\tilde{y})\,d\tilde{x}\,d\tilde{y}$$

is the solution to the following boundary value problem for the heat equation:

$$(\partial_s + \Box_b^\lambda)\{H^\lambda f\} = 0, \qquad H^\lambda\{f\}(s=0,x,y) = f(x,y).$$

5.4 Group Convolution and Twisted Convolution

To motivate the above formula, we consider the fundamental solution to the (full) unweighted heat equation: $\rho_s(x,y,t)$. For a function $f_0 \in L^2(\mathbb{C}^n \times \mathbb{R}^m)$, and $g = (z,t) \in \mathbb{C}^n \times \mathbb{R}^m$, define

$$H\{f_0\}(s,g) = (\rho_s * f_0)(g) = \int_{\tilde{g}} \rho_s(g[\tilde{g}]^{-1})f_0(\tilde{g})\,d\tilde{g} \tag{21}$$

where $*$ is the group convolution and $g[\tilde{g}]^{-1}$ is the group multiplication of g by the inverse of $\tilde{g}$. If X is a right invariant vector field, then

$$XH\{f_0\}(s,g) = \int_{\tilde{g}} (X\rho_s)(g[\tilde{g}]^{-1})f_0(\tilde{g})\,d\tilde{g}.$$

Since $\Box_b$ is composed of right invariant vector fields and ρ_s satisfies the $\Box_b$-heat equation, we therefore have

$$(\partial_s + \Box_b)\{H(f_0)\} = 0.$$

In addition, the following initial condition holds:

$$H\{f_0\}(s=0,g) = \int_{\tilde{g}} \rho_{s=0}(g[\tilde{g}]^{-1})f_0(\tilde{g})\,d\tilde{g} = f_0(g)$$

since $\rho_{s=0}(z,t)$ is the Dirac delta function centered at $(z,t) = 0$.

Note that $H^\lambda\{f\}(s,x,y) = (2\pi)^{m/2}H\{f\}(s,x,y,\widehat{\lambda})$, which is the partial Fourier transform in the t variable of $H\{f\}(s,x,y,t)$. We will now show the Fourier transform in the t-variable transforms the group convolution to a "twisted convolution", which we now define. Suppose F and G are in $L^2(\mathbb{C}^n)$, and $\lambda \in \mathbb{R}^m$. Following Stein [18], p. 552, we let

$$(F *_\lambda G)(z) = \int_{\tilde{z}\in\mathbb{C}^n} F(z-\tilde{z})G(\tilde{z})e^{-2i\lambda\cdot \operatorname{Im}\phi(z,\tilde{z})}\,d\tilde{z}.$$

The arguments in [18], p. 552, with $\langle z,\tilde{z}\rangle$ replaced by $2\operatorname{Im}\phi(z,\tilde{z})$, show the following: if $F_0, G_0 \in L^2(\mathbb{C}^n\times\mathbb{R}^m)$, then

$$(F_0 * G_0)(z,\widehat{\lambda}) = (2\pi)^{m/2}(F_0(\cdot,\widehat{\lambda}) *_\lambda G_0(\cdot,\widehat{\lambda}))(z).$$

Now suppose $f \in L^2(\mathbb{C}^n)$ is given and let $f_0(z,t) = (2\pi)^{m/2}f(z)\delta_0(t)$, so that $f_0(z,\widehat{\lambda}) = f(z)$. With H^λ given as in (20), we can take the partial Fourier transform in t of (21) and use the above relationship to obtain

$$\begin{aligned} H^\lambda(f)(s,z) &= \int_{\tilde{z}\in\mathbb{C}^n} H^\lambda(s,z,\tilde{z})f(\tilde{z})\,d\tilde{z} \\ &= \int_{\tilde{z}\in\mathbb{R}^n,\tilde{y}\in\mathbb{R}^n} (2\pi)^{m/2}\rho_s(x-\tilde{x},y-\tilde{y},\widehat{\lambda})f(\tilde{x},\tilde{y})e^{-2i\lambda\cdot\operatorname{Im}\phi(z,\tilde{z})}\,d\tilde{x}\,d\tilde{y} \\ &= (2\pi)^{m/2}(\rho_s(\cdot,\widehat{\lambda}) *_\lambda f_0(\cdot,\widehat{\lambda}))(z) \\ &= (\rho_s * f_0)(z,\widehat{\lambda}) \\ &= H(f_0)(s,z,\widehat{\lambda}). \end{aligned}$$

Since $H(f_0)$ satisfies the $\Box_b$-heat equation, $H(f_0)(s,z,\widehat{\lambda}) = H^\lambda(f)(s,z)$ satisfies the weighted heat equation, i.e.,

$$(\partial_s + \Box_b^\lambda)\{H^\lambda(f)\} = 0.$$

The initial condition $H^\lambda(f)(s=0,z) = f(z)$ is also satisfied because

$$\begin{aligned} H^\lambda(f)(s=0,z) &= H(f_0)(s=0,z,\widehat{\lambda}) \\ &= f_0(z,\widehat{\lambda}) \\ &= f(z). \end{aligned}$$

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