HEAT EQUATIONS IN $\mathbb{R} \times \mathbb{C}$

By

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Abstract

Let $p : \mathbb{C} \to \mathbb{R}$ be a subharmonic, nonharmonic polynomial and $\tau \in \mathbb{R}$ a parameter. Define $\bar{Z}_{\tau p} = \frac{\partial}{\partial \bar{z}} + \tau \frac{\partial p}{\partial \bar{z}}$, a closed, densely-defined operator on $L^2(\mathbb{C})$. If $\Box_{\tau p} = \bar{Z}_{\tau p} \bar{Z}^*_{\tau p}$ and $\widetilde{\Box}_{\tau p} = \bar{Z}^*_{\tau p} \bar{Z}_{\tau p}$, the goal of this thesis is to solve the heat equations: $\frac{\partial u}{\partial s} + \Box_{\tau p} u = 0$, $u(0, z) = f(z)$ and $\frac{\partial \tilde{u}}{\partial s} + \Box_{\tau p} \tilde{u} = 0$, $\tilde{u}(0, z) = \tilde{f}(z)$ on $(0, \infty) \times \mathbb{C}$. The solutions come via the heat semigroups $e^{-s\Box_{\tau p}}$ and $e^{-s\widetilde{\Box}_{\tau p}}$, and we show that $u(s, z) = e^{-s\Box_{\tau p}}[f](z) = \int_{\mathbb{C}} H_{\tau p}(s, z, w) f(w)\, dw$ and $\tilde{u}(s, z) = e^{-s\widetilde{\Box}_{\tau p}}[\tilde{f}](z) = \int_{\mathbb{C}} \tilde{H}_{\tau p}(s, z, w) \tilde{f}(w)\, dw$. We prove that $H_{\tau p}, \tilde{H}_{\tau p}$ are C^∞ off the diagonal $\{(s, z, w) : s = 0 \text{ and } z = w\}$ and that $H_{\tau p}$ and its derivatives have exponential decay. We develop classes of one-parameter families (OPF) of operators on $C^\infty_c(\mathbb{C})$ which are instrumental in proving both the regularity of $H_{\tau p}$ and $\tilde{H}_{\tau p}$ and the decay of $H_{\tau p}$. We prove that an order 0 OPF operator extends to a bounded operator from $L^q(\mathbb{C})$ to itself, $1 < q < \infty$, with a bound that depends on q and the degree of p but not on τ or the coefficients of p. Last, we show that there is a one-to-one correspondence given by the partial Fourier transform in τ between OPF operators of order $m \leq 2$ and nonisotropic smoothing (NIS) operators of order $m \leq 2$ on polynomial model domains in $\mathbb{C}^2$.

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Chapter 1

Introduction

1.1 Introduction

Let $p : \mathbb{C} \to \mathbb{R}$ be a subharmonic, nonharmonic polynomial. If $z = x_1 + ix_2$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right)$, define $\bar{Z}_p$ to be the operator

$$\bar{Z}_p = \frac{\partial}{\partial \bar{z}} + \frac{\partial p}{\partial \bar{z}},$$

and let $Z_p = -(\bar{Z}_p)^* = \frac{\partial}{\partial z} - \frac{\partial p}{\partial z}$ be the negative of the formal L^2-adjoint of $\bar{Z}_p$. If we let $\Box_p = -\bar{Z}_p Z_p$ and $\widetilde{\Box}_p = -Z_p\bar{Z}_p$, then the research in this thesis is focused on understanding the heat equations:

$$\begin{cases} \dfrac{\partial u}{\partial s} + \Box_p u = 0 \\ \\ u(0, z) = f(z) \end{cases} \tag{1.1}$$

and

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial s} + \widetilde{\Box}_p \tilde{u} = 0 \\ \\ \tilde{u}(0, z) = \tilde{f}(z). \end{cases} \tag{1.2}$$

Our goal is show that the solutions u and $\tilde{u}$ of (1.1) and (1.2), respectively, can be realized as integrals against distributional kernels. Specifically, we want a solution of

(1.1) of the form:

$$u(s,z) = \int_{\mathbb{C}} H(s,z,w) f(w)\, dw$$

and a solution of (1.2) of the form:

$$\tilde{u}(s,z) = \int_{\mathbb{C}} \tilde{H}(s,z,w) \tilde{f}(w)\, dw.$$

We are interested in showing that H and $\tilde{H}$ are smooth (on an appropriate region) and establishing pointwise estimates for the kernels.

The operators $\bar{Z}_p$ and Z_p arise is two natural ways. One is through the study of $\bar{\partial}_b$ on a class of weakly pseudoconvex domains of finite type called polynomial model domains, and the other is through the study of the weighted $\bar{\partial}$-equation in $\mathbb{C}$. In both cases, $\bar{Z}_p$ occurs during an investigation of the $\bar{\partial}$-problem on domains $\Omega \subset \mathbb{C}^n$, which is where we begin our discussion.

1.2 $\bar{\partial}$-equation on Ω

Let $\Omega \subset \mathbb{C}^n$ be an open set. Given a function $u : \Omega \to \mathbb{C}$, the Cauchy-Riemann operator $\bar{\partial}$ acting on u is defined by

$$\bar{\partial} u = \sum_{j=1}^{n} \frac{\partial u}{\partial \bar{z}_j} d\bar{z}_j.$$

In complex analysis, an important problem analysis is to solve the inhomogeneous Cauchy-Riemann equations on Ω. This means given $f(z) = \sum_{j=1}^{n} a_j(z)\, d\bar{z}_j$, finding a function u so that $\bar{\partial} u = f$. This is a difficult problem because the system is over-determined, i.e. there are more equations than unknowns. One requirement to solve $\bar{\partial} u = f$ is the compatability condition $\bar{\partial} f = 0$ where $\bar{\partial} f = \sum_{j=1}^{n} \bar{\partial} a_j(z) \wedge d\bar{z}_j$. In general, when trying to solve any partial differential equation or system of partial differential equations, it is important to determine existence and uniqueness of solutions. On Ω, there exist functions h so that $\bar{\partial}h = 0$. Such function are called *holomorphic*, and the study of holomorphic functions is a major focus of complex analysis. The existence of holomorphic functions means that there is no uniqueness for the $\bar{\partial}$-problem, $\bar{\partial}u = g$. However, this does not mean that there is no hope of solving the $\bar{\partial}$-problem on Ω. One must ask a question which, if there is an answer, guarantees uniqueness. One possibility is to find the L^2-minimizing solution, and this is the approach many authors take. The goal becomes not just solving a particular PDE but solving with "estimates". This means understanding the mapping properties of solutions, i.e. understanding the smoothness or size of the solution relative to the size of the data or initial condition.

There are a number of classes of domains on which the $\bar{\partial}$-problem is studied, and the class on which we work is the class of is pseudoconvex domains. One reason pseudoconvex domains are studied is that if Ω is pseudoconvex, there exists a holomorphic function on Ω which cannot be holomorphically extended to a larger domain. In a sense, pseudoconvex domains form the maximal domains in $\mathbb{C}^n$ on which holomorphic functions are defined. Kohn and Hörmander pioneered the early work on the $\bar{\partial}$-problem [Koh61, Koh63, Koh64, Hör65], and Hörmander's methods on pseudoconvex domains, now classical in the subject, rely on proving estimates in weighted L^2-spaces [Hör90, Kra01]. This motivates studying the $\bar{\partial}$-problem on the weighted space $L^2(\mathbb{C}, e^{-2p})$.

1.3 $\bar{\partial}$ on Weighted L^q Spaces in $\mathbb{C}$

Let p be subharmonic but not harmonic. On $\mathbb{C}$, $\bar{\partial}u = \frac{\partial u}{\partial \bar{z}}\, d\bar{z}$. Since there is only one term in the sum, we identify $\bar{\partial}u$ with $\frac{\partial u}{\partial \bar{z}}$ and omit the $d\bar{z}$. When $\Omega \subset \mathbb{C}$ is bounded, Hörmander shows that there is a solution u to

$$\bar{\partial}u = f \tag{1.3}$$

in $L^2(\Omega, e^{-2p})$ satisfying the estimate:

$$\int_\Omega |u|^2 e^{-2p}\, dz \leq (\operatorname{diam}\Omega)^2 \int_\Omega |f|^2 e^{-2p}\, dz.$$

If $\operatorname{diam}\Omega < 1$, Fornæss and Sibony [FS91] generalize this estimate to L^q, $1 < q \leq 2$, and prove that (1.3) has a solution satisfying:

$$\left(\int_\Omega |u|^q e^{-2p}\, dz\right)^{\frac{1}{q}} \leq \frac{C}{p-1}\left(\int_\Omega |f|^q e^{-2p}\, dz\right)^{\frac{1}{q}}.$$

They also show that the estimate fails if $q > 2$. Berndtsson [Ber92] builds on the work of Fornæss and Sibony by showing an L^q-L^1 result. He shows that if $\operatorname{diam}\Omega < 1$ and $1 \leq q < 2$, then (1.3) has a solution so that

$$\left(\int_\Omega (|u|^2 e^{-p})^q\, dz\right)^{\frac{1}{q}} \leq C_p \int_\Omega |f| e^{-p}\, dz.$$

Berndtsson also proves a weighted L^∞-L^q estimate when $q > 2$, but the estimate is more complicated to write down.

In [Chr91a], Christ recognizes that it is possible to study the $\bar{\partial}$-problem in $L^2(\mathbb{C}, e^{-2p})$ by working with a related operator in the unweighted space $L^2(\mathbb{C})$. If $\bar{\partial}\tilde{u} = \tilde{f}$ and both $\tilde{u} = e^p u$ and $\tilde{f} = e^p f$ are in $L^2(\mathbb{C}, e^{-2p})$, then

$$\frac{\partial(e^p u)}{\partial \bar{z}} = e^p f \Longrightarrow e^{-p}\frac{\partial}{\partial \bar{z}} e^p u = f.$$

However, $e^{-p}\frac{\partial}{\partial \bar{z}}e^{p}u = \bar{Z}_p u$, so the $\bar{\partial}$-problem on $L^2(\mathbb{C}, e^{-2p})$ is equivalent to the $\bar{Z}_p$-problem, $\bar{Z}_p u = f$, on $L^2(\mathbb{C})$. Christ solves the $\bar{Z}_p$-equation $\bar{Z}_p u = f$. In $L^2(\mathbb{C})$, the null space of $\bar{Z}_p$ is large, so Christ finds the the L^2-minimizing solution. Christ proves that $G_p = \Box_p^{-1}$ is a well-defined, bounded, linear operator on $L^2(\mathbb{C})$. $R_p = Z_p G_p$ is the relative fundamental solution of $\bar{Z}_p$, i.e. the operator R_p satisfies $\bar{Z}_p R f = (I - S_p)f$ where S_p is the projection of $L^2(\mathbb{C})$ onto the $\ker \bar{Z}_p$. It is also the L^2-minimizing solution since $\text{Range}(Z_p) = (\ker \bar{Z}_p)^{\perp}$. He shows that G_p and R_p can be realized as fractional integral operators with kernels $G_p(z, w)$ and $R_p(z, w)$ respectively. This means that

$$G_p[f](z) = \int_{\mathbb{C}} G_p(z, w) f(w)\, dw$$

and

$$R_p[f](z) = \int_{\mathbb{C}} R_p(z, w) f(w)\, dw$$

where $G_p(z, w)$ and $R_p(z, w)$ are distributions with integrable singularities on $\{z = w\}$. Christ obtains pointwise upper bounds on both the blow up of the singularities on the diagonal $\{z = w\}$ and the decay at infinity. A difficulty in analyzing $\Box_p$ is that it is hard to solve $\Box_p u = f$ directly. Instead, Christ establishes good local L^2 decay estimates for solutions and proves a local L^∞-L^2 type bound on local solutions of $\Box_p u = 0$. Specifically, for the local L^∞-L^2 type bound, he shows that if $\Box_p u = 0$ on $D = D(z_0, r)$, then there exists N, C so that $\|u\|_{L^\infty(D)} + r\|Z_p u\|_{L^\infty(D)} \leq C(1 + \nu(D))^N (r^{-1}\|u\|_{L^2(2D)} + \|Z_p u\|_{L^2(2D)})$ where $d\nu = \triangle p\, dz$.

In [Ber96], Berndtsson also solves $\bar{Z}_p u = f$ for p subharmonic, but Berndtsson solves the problem on $L^2(\Omega)$ where $\Omega \subset \mathbb{C}$ is a smoothly bounded domain. Like Christ, he expresses his L^2-minimizing solution via a fractional integral operator, though unlike

Christ, his analysis is derived through functional analysis and a careful study of Kato's inequality:

$$\triangle|\alpha| \geq \triangle p|\alpha| - 4|\Box_p \alpha|$$

for $\alpha \in C^2(\Omega)$. If R_p^Ω is the relative fundamental solution for $\bar{Z}_p$ on Ω with kernel $R_p^\Omega(z,\zeta)$, and R_0^Ω is the relative fundamental solution for $\bar{\partial}$ (in $L^2(\mathbb{C})$) with kernel $R_0^\Omega(z,\zeta)$, Berndtsson proves

$$|R_p^\Omega(z,\zeta)| \leq |R_0^\Omega(z,\zeta)|$$

for $z \in \partial\Omega$. He also shows if p and ρ are subharmonic, $\Omega = \{z : \rho(z) < 0\}$, and $\left|\frac{\partial\rho}{\partial z}\right| \neq 0$ on $\partial\Omega$, then if $|f| \leq (-\rho)\triangle p$ and $u(z) = R_p^\Omega[f](z)$,

$$|u| \leq \left|\frac{\partial \rho}{\partial z}\right|$$

on $\partial\Omega$.

1.4 Pseudoconvex Domains and $\bar{\partial}_b$

Now that we have established the connection between the weighted $\bar{\partial}$-equation in $\mathbb{C}$ and the operators $\bar{Z}_p$ and Z_p, we now turn to the study of pseudoconvex domains and the $\bar{\partial}_b$-problem, and their connection with the operators $\bar{Z}_p$ and Z_p.

A *defining function* ρ for Ω is a function $\rho : \mathbb{C}^n \to \Omega$ so that $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0 \text{ and } \nabla\rho \neq 0 \text{ when } \rho = 0\}$. We say that $\partial\Omega$ is C^k if ρ is C^k on a neighborhood of $\partial\Omega$. When $\partial\Omega$ is C^2, we can write down a geometric condition on $\partial\Omega$ to describe pseudoconvexity. Given $z \in \partial\Omega$, the *real tangent space* at z is $T_z(\partial\Omega) = \{\xi = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} : \xi \cdot \nabla\rho = 0\}$. We identify $\mathbb{R}^{2n} \cong \mathbb{C}^n$ with the identification $\xi = (x_1 + ix_{n+1}, \ldots, x_n + ix_{2n}) = (\xi_1, \ldots, \xi_n)$. Under this identification, if

$\frac{\partial}{\partial \xi_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_{n+j}}\right)$, then

$$\xi \cdot \nabla\rho = 0 \Longleftrightarrow \mathrm{Re}\left(\sum_{j=1}^{n} \frac{\partial\rho}{\partial\xi_j}\xi_j\right) = 0,$$

and it follows that the *maximal complex tangent space* at z is

$$T_z^{\mathbb{C}}(\partial\Omega) = \left\{\xi \in \mathbb{C}^n : \sum_{j=1}^{n} \frac{\partial\rho}{\partial\xi_j}\xi_j = 0\right\}.$$

The complex Hession of ρ, $(\frac{\partial^2\rho}{\partial z_i \partial\bar{z}_j})$, is called the Levi form of ρ, and pseudoconvexity is equivalent to the nonnegativity of the Levi form on $T_z^{\mathbb{C}}(\partial\Omega)$. One comment about "the" Levi form is that different defining functions for Ω will give different Levi forms. However, the signs of the eigenvalues of the Levi form are invariant under biholomorphic changes of coordinates.

One reason having a characterization of pseudoconvexity directly in terms the boundary data is desirable is that the geometry of the boundary determines whether or not holomorphic functions extend to the boundary and in what sense they extend. We know there exist holomorphic functions on Ω which do not extend holomorphically across $\partial\Omega$, but it may be possible to extend the function to a space of distributions or a function space defined on $\partial\Omega$. $\partial\Omega$ is a submanifold of $\mathbb{C}^n$ of real codimension 1. A related problem to studying the extension of holomorphic functions to $\partial\Omega$ is to take a smooth submanifold $M \subset \mathbb{C}^n$ and study the equations that holomorphic functions must satisfy when restricted to M. Phrased differently, if we are given a function on M, we would like to know if that function can be the restriction of a holomorphic function. When $n \geq 2$, these questions are naturally answered in the language of complex vector fields.

It is standard in the subject to identify the vector $(a_1, \ldots, a_{2n}) \in \mathbb{R}^{2n}$ with the vector $\sum_{j=1}^{2n} a_j \frac{\partial}{\partial x_j} = a \cdot \nabla$. Let $U \subset \mathbb{R}^{2n}$ be an open set, and if $z \in U$, let T_zU be the space of tangent vectors at U. We define the *tangent bundle of* U as $T(U) = \bigcup_{z\in U} T_z(U)$ and a *vector field on* U as a map $X : U \to T(U)$ defined by $X = \sum_{j=1}^{2n} a_j(z)\frac{\partial}{\partial x_j}$ where $a_j \in C^\infty(U)$ and $\mathbb{R}$-valued. For our purposes, we would like to allow $a_j(z)$ to be $\mathbb{C}$-valued, so we define the *complexified tangent bundle of* U by $T(U)\otimes\mathbb{C}$. A *complex vector field* is a map $X : U \to T(U)\otimes\mathbb{C}$ where $X = \sum_{j=1}^{2n} a_j(z)\frac{\partial}{\partial x_j}$ and $a_j \in C^\infty(U)$ are $\mathbb{C}$-valued. For the rest of the exposition, we will assume that all vector fields are complex vector fields. A vector field X which satisfies $X\rho = 0$ is called a *tangential vector field.* The name tangential follows from the equivalence $X\rho = 0$ if and only if $X \perp \nabla\rho$. Earlier, we discussed the complex structure on $\mathbb{C}^n$, and if we let $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial \bar{x}_j} + \frac{\partial}{\partial \bar{x}_{n+j}}\right)$, then there is the decomposition $T(U)\otimes\mathbb{C} = T^{(1,0)}U \oplus T^{(0,1)}U$, where

$$T^{(1,0)}U = \left\{ L \in T(M)\otimes\mathbb{C} : L = \sum_{j=1}^{n} a_j(z)\,\frac{\partial}{\partial z_j} \right\},$$

and

$$T^{(0,1)}U = \overline{T^{(1,0)}U} = \left\{ \bar{L} \in T(M)\otimes\mathbb{C} : \bar{L} = \sum_{j=1}^{n} b_j(z)\,\frac{\partial}{\partial \bar{z}_j} \right\}.$$

Note that holomorphic functions on U are exactly the functions annihilated by vector fields of the form $\bar{L} = \sum_{j=1}^{n} b_j(z)\frac{\partial}{\partial \bar{z}_j}$. Such vector fields are called *antiholomorphic vector fields.*

To apply a vector field X defined on Ω to a function g on M, we extend g to G on Ω and define $Xg = XG\big|_M$. One obvious problem with this idea is that it is not clear whether Xg is well-defined. It turns out that Xg is well-defined if and only if $X\rho = 0$. We can now answer questions posed about holomorphic functions on M. The linearly independent set $\{\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\}$ generates the module of antiholomorphic vector fields on Ω over $C^\infty(\Omega)$. We denote the dual of the tangent vector $\frac{\partial}{\partial \bar{z}_j}$ by the $(0,1)$-form $d\bar{z}_j$. A

$(0,1)$-form is a differential form of the form $\alpha = \sum_{j=1}^{n} \alpha_j(z)\, d\bar{z}_j$ where $\alpha_j \in C^\infty(\Omega)$. We know the condition $\bar{L}\rho = 0$ is necessary to define $\bar{L}g$, so locally there is a family of $(n-1)$ linearly independent antiholomorphic tangential vector fields $\bar{L}_j$, $1 \le j \le n-1$. For $z \in M$, $\{\bar{L}_1(z), \ldots, \bar{L}_{n-1}(z)\}$ forms a basis of the tangential antiholomorphic vector fields in $T_z^{\mathbb{C}}(M)$, and let $\{d\bar{\omega}_1(z), \ldots, d\bar{\omega}_{n-1}(z)\}$ be the dual basis of $(0,1)$-forms. Analogously to the $\bar{\partial}$-operator on Ω, we define the boundary Cauchy-Riemann operator $\bar{\partial}_b$ on the function g defined on M as

$$\bar{\partial}_b g = \sum_{j=1}^{n-1} \bar{L}_j[g]\, d\bar{\omega}_j.$$

If $\bar{\partial}_b g = 0$ and Ω satisfies suitable hypotheses, we can extend g holomorphically to Ω.

We are interested in questions on $M = \partial\Omega \subset \mathbb{C}^2$ related to the $\bar{\partial}_b$-problem, $\bar{\partial}_b g = f$, when Ω is pseudoconvex. The geometry of M plays a vital role in the tractibility of solving the $\bar{\partial}_b$-problem. We can understand the geometry of M by understanding the ρ and its Levi form. $T_z^{\mathbb{C}}(M) = \{(u_1, u_2) \in \mathbb{C}^2 : \frac{\partial\rho}{\partial z_1} u_1 + \frac{\partial\rho}{\partial z_2} u_2 = 0\}$, a one complex dimensional space with basis vector $(-\frac{\partial\rho}{\partial z_2}, \frac{\partial\rho}{\partial z_1})$. Thus, if $\lambda, \mu \in \mathbb{C}$ are nonzero, then $\lambda(-\frac{\partial\rho}{\partial z_2}, \frac{\partial\rho}{\partial z_1})$ and $\mu(-\frac{\partial\rho}{\partial z_2}, \frac{\partial\rho}{\partial z_1})$ are arbitrary nonzero vectors in $T_z^{\mathbb{C}}(M)$, and the Levi form at z is written

$$\bar{\mu}\lambda \begin{pmatrix} -\dfrac{\partial\rho(z)}{\partial z_2} & \dfrac{\partial\rho(z)}{\partial z_1} \end{pmatrix} \begin{pmatrix} \dfrac{\partial^2\rho(z)}{\partial z_1 \partial \bar{z}_1} & \dfrac{\partial^2\rho(z)}{\partial z_1 \partial \bar{z}_2} \\ \dfrac{\partial^2\rho(z)}{\partial z_2 \partial \bar{z}_1} & \dfrac{\partial^2\rho(z)}{\partial z_2 \partial \bar{z}_2} \end{pmatrix} \begin{pmatrix} -\dfrac{\partial\rho(z)}{\partial \bar{z}_2} \\ \dfrac{\partial\rho(z)}{\partial \bar{z}_1} \end{pmatrix} = \bar{\mu}\lambda L(z).$$

If $L(z) > 0$, the Levi form at z is positive definite, and we say that the Levi form is *strictly pseudoconvex* at z. If $L(z) = 0$, then the Levi form is nonnegative at z, and we say that the Levi form is *weakly pseudoconvex* at z. At z, if m is the lowest order derivative of L that is nonzero, then z is a point of *type m*. If M has points of type m and lower, M is said to be of *finite type*. In $\mathbb{C}^2$, there has been progress on the $\bar{\partial}_b$ problem when Ω is bounded and strictly pseudoconvex. In particular, there has been analysis both of the *Szegö projection* $S : L^2(M) \to \ker \bar{\partial}_b$ and of the relative fundamental solution R of $\bar{\partial}_b$, i.e. the operator R satisfies $\bar{\partial}_b R f = (I - S) f$. Similarly to Christ's analysis of $\bar{Z}_p$, $\bar{\partial}_b$ is analyzed via the operator $\Box_b = -\bar{\partial}_b^* \bar{\partial}_b$ which has relative fundamental solution G. To discuss the $\mathbb{C}^2$ results, it is helpful to introduce a class of operators called nonisotropic smoothing (NIS) operators defined in [NRSW89]. Given a manifold $N \subset \mathbb{R}^n$, an NIS operator T acting on functions $\varphi \in C_c^\infty(N)$ can be written as

$$T[\varphi](\alpha) = \int_N K(\alpha, \beta) \varphi(\beta) \, d\beta$$

for some distributional kernel $K(\alpha, \beta)$ that is smooth off of the diagonal $\alpha = \beta$. Also, the behavior of T is governed by a size condition on K and a cancellation condition governing the size of $\|T[\varphi]\|_{L^\infty}$. The final condition is one that makes NIS operators into an algebra. NIS operators can be viewed as analogs to Calderòn-Zygmund operators where the size is governed by a metric defined by the tangential vectors fields instead of the Euclidean distance. The results about S, R, and G can be summarized as follows: S, R, and G are NIS operators of orders 0, 1, and 2, respectively. These results are the culmination of the work of many authors, including Christ [Chr88b, Chr88a, Chr91b, Chr91a], Fefferman and Kohn [FK88a, FK88b], Kohn [Koh72, Koh85], McNeal [McN89], and Nagel, Rosay, Stein, and Wainger [NRSW89], See Christ[Chr91c] for a helpful exposition regarding timeline of the papers and concise statements of the results. See Fefferman [Fef95] for a discussion of the logic behind the arguments. There has also been progress for a class of unbounded weakly pseudoconvex domains of finite type called polynomial model domains [NRSW89]. These are domains of the form

$$M = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 = p(z_1)\}.$$

where p is a subharmonic, nonharmonic polynomial. We now restrict our attention to these domains.

Our focus is on questions related to the $\bar{\partial}_b$-problem on polynomial model domains in $\mathbb{C}^2$. Observe that $M \cong \mathbb{C} \times \mathbb{R}$ under the isomorphism

$$(z_1, z_2) = \big(z, t + ip(z)\big) \mapsto (z, t).$$

On hypersurfaces in $\mathbb{C}^2$, the module of tangential, antiholomorphic vector fields over C^∞ is spanned by one element. On M, we can take

$$\bar{\partial}_b = \bar{L}\, d\bar{\omega} = \left(\frac{\partial}{\partial \bar{z}_1} - 2i\frac{\partial p}{\partial \bar{z}_1}\frac{\partial}{\partial \bar{z}_2}\right)\, d\bar{\omega}.$$

Because there is only one term, we suppress the $d\bar{\omega}$ term and identify $\bar{\partial}_b$ with the vector field $\bar{L}$. We write write

$$\bar{\partial}_b = \bar{L} = \frac{\partial}{\partial \bar{z}_1} - 2i\frac{\partial p}{\partial \bar{z}_1}\frac{\partial}{\partial \bar{z}_2}$$

Under the isomorphism, $\bar{\partial}_b$, defined on M, becomes the vector field (still called $\bar{L}$ by an abuse of notation)

$$\bar{L} = \frac{\partial}{\partial \bar{z}} - i\frac{\partial p}{\partial \bar{z}}\frac{\partial}{\partial t}$$

defined on $\mathbb{C} \times \mathbb{R}$. There are a number of approaches that one can take to study the $\bar{L}$-problem. One is to observe that $\bar{L}$ is translation invariant in t, and this suggests that we take a partial Fourier transform in t. If $f(z,t)$ is a suitably nice function on $\mathbb{C} \times \mathbb{R}$, the *partial Fourier transform* of f is

$$F(z, \tau) = \int_{\mathbb{R}} e^{-it\tau} f(z,t)\, dt.$$

Under the partial Fourier transform, the vector field $\bar{L}$ becomes $\bar{Z}_{\tau p} = \frac{\partial}{\partial \bar{z}} + \tau\frac{\partial p}{\partial \bar{z}}$, which we regard as a one-parameter family of differential operators on $\mathbb{C}$ indexed by τ. As discussed earlier, there is a strong connection between the operators $\bar{Z}_{\tau p}$ and the $\bar{\partial}$-equation on weighted L^2-spaces in $\mathbb{C}$. Thus, questions about the $\bar{\partial}_b$-complex on M are intimately connected with the $\bar{\partial}$-equation on weighted L^2-spaces on $\mathbb{C}$.

1.5 $\Box_{\tau p}$-Heat Equations and OPF Operators

Like Christ, we are interested in inverting $\Box_{\tau p}$. To study $\Box_{\tau p}$, Christ's methods are not the only ones available. For a different approach to invert $\Box_{\tau p}$, we can look at the heat semigroup $e^{-s\Box_{\tau p}}$ and integrate out s. Formally, $u = e^{-s\Box_{\tau p}}[f]$ solves the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial s} + \Box_{\tau p} u = 0 \\ \\ u(0, z) = f(z) \end{cases} \tag{1.4}$$

and inverts $\Box_{\tau p}$ since

$$\int_0^\infty e^{-s\Box_{\tau p}}\, ds = \Box_{\tau p}^{-1}. \tag{1.5}$$

Nagel and Stein [NS01] investigate the heat semigroup $e^{-s\Box_b}$ on M. Their goal is to use estimates on the heat semigroup on $M \cong \mathbb{C}\times\mathbb{R}$ to understand $\Box_b$ in a product setting [NS04]. Nagel and Stein use the spectral theorem to define $e^{-s\Box_b}$ as a distributional kernel and prove that it is a contraction on $L^2(\mathbb{C}\times\mathbb{R})$. Next, they use the Riesz Representation Theorem to justify writing

$$e^{-s\Box_b}[f](\alpha) = \int_{\mathbb{C}\times\mathbb{R}} H(s, \alpha, \beta) f(\beta)\, d\beta,$$

where H is a distributional kernel with a nonintegrable singularity when $s = 0$ and $p = q$. They then show the heat kernel $H(s, \alpha, \beta)$ and its derivatives are actually smooth off of the diagonal and obtain estimates on $H(s, \alpha, \beta)$ and its derivatives. It turns out that an analog to (1.5) cannot hold because $\ker \Box_b \neq \{0\}$, but a substitute formula works; if S is the Szegö projection, i.e. the projection of $L^2(M)$ onto $\ker \Box_b$, then

$$\int_0^\infty e^{-s\Box_b}(I-S)\,ds = \Box_b^{-1}$$

where $\Box_b^{-1}$ stands for the L^2-minimizing inverse operator.

In their argument to prove smoothness, NIS operators play an instrumental role. The spectral theorem gives control of certain derivatives but not all derivatives. Here, derivative means the operator $\bar{L}$ or $\bar{L}^*$. Roughly speaking, if α is a multiindex and Y^α is a product of $|\alpha|$ operators of the form $Y = \bar{L}$ or $\bar{L}^*$, then results from NIS operator theory allows Nagel and Stein to write

$$Y^\alpha = A\Box_b^{\frac{|\alpha|}{2}} \tag{1.6}$$

where A is an NIS operator that is well controlled. Switching from Y^α to $A\Box_b^{\frac{|\alpha|}{2}}$ is analogous to using the Riesz transform to switch from arbitrary derivatives to powers of the Laplacian. This is an extremely usefal fact because the spectral theorem contains the estimate $\|\Box_b^j\varphi\|_{L^2(\mathbb{C}\times\mathbb{R})} \lesssim s^{-j}\|\varphi\|_{L^2(\mathbb{C}\times\mathbb{R})}$, and (1.6) allows the estimate to be extended to $\|Y^\alpha\varphi\|_{L^2(\mathbb{C}\times\mathbb{R})} \lesssim s^{-|\alpha|/2}\|\varphi\|_{L^2(\mathbb{C}\times\mathbb{R})}$. The size estimate that Nagel and Stein prove is shown using a scaling argument. The structure of $\Box_b$ allows a reduction from finding pointwise estimates of $H(s,\alpha,\beta)$ at arbitrary points α and β from an arbitrary polynomial model domain to finding pointwise estimates of kernels $H(s,0,\beta)$ where β and s are a unit distance from 0, and the polynomial which defines M is from a compact set of normalized polynomials.

A motivation for this thesis is an attempt to solve the problem of Christ, i.e. invert $\Box_{\tau p}$ and find pointwise estimates on $G_{\tau p}(z,\zeta)$ and its derivatives, using the heat semigroup $e^{-s\Box_{\tau p}}$ method motivated by Nagel and Stein. In addition, understanding the heat equation (1.4) is an interesting question in its own right. There are a number of obstructions to using the techniques of Nagel and Stein in this setting. First, there is no analog to NIS operators, so we define classes of one-parameter families (OPF) of operators to play an analogous role to NIS operators. Second, due to the partial Fourier transform, it appears that we cannot scale in the transformed variable. Losing the ability to scale in any variable dooms the scaling argument of Nagel and Stein. We find other techniques which allow us to bound the heat kernel and its derivatives with better decay than the scaling argument would have given.

To define the OPF operators, the operators of Christ form the starting point. We define classes of OPF operators that act on $C_c^\infty(\mathbb{C})$ based on the operatrs G_p and R_p. Similarly to both [NRSW89] and [NS01], we spend considerable effort exploring the properties of our new class of operators. Since OPF operators were to be analogs of NIS operators on polynomial model domains, we knew that OPF operators as a class would have useful properties that individual operators would not. Indeed, the class of OPF operators is closed under translation, rotation, dilation, and composition. This enables us to use a scaling argument to help prove the decay of the heat kernel. As with NIS operators, another useful property of the OPF operators is if α is a multiindex and $Y = Z_{\tau p}$ or $\bar{Z}_{\tau p}$, then, roughly speaking, we can write $Y^\alpha = A_\tau \Box_{\tau p}^{\frac{|\alpha|}{2}}$ where A_τ is a well controlled OPF operator.

To motivate the definition of an OPF operator, it is important to recall that OPF operators were designed to be analogs of NIS operators. In fact, one of the original motivations for defining OPF operators was to study how an NIS operator on a polynomial model domain in $\mathbb{C}^2$ behaves under a partial Fourier transform.

Qualitatively, a partial Fourier transform on an NIS operator will not change the fact that there is an integral kernel, size conditions, and cancellation conditions. Any reasonable definition of an OPF operator will have to have a size and cancellation condition and an integral kernel. Specifically, given a subharmonic, nonharmonic polynomial p, a one-parameter family of operators T_τ of order $m \leq 2$ acts on $\varphi \in C_c^\infty(\mathbb{C})$ by

$$T_\tau[\varphi](z) = \int_{\mathbb{C}} K_\tau(z,w)\varphi(w)\,dw$$

where K_τ is a distributional kernel that is C^∞ away from the diagonal $\{z = w\} \times \{\tau = 0\}$. If $(Y_{\tau p})^J$ is a product of $\bar{Z}_{\tau p}$ and $Z_{\tau p}$ of length $|J| = \ell$, there exists a constant $C_{\ell,k}$ so that there is the size condition

$$\left|Y_{\tau p}^J K_\tau(z,w)\right| \leq C_{\ell,k} \frac{|z-w|^{2-m-\ell}}{\tau^k \Lambda(z,|w-z|)^k} \quad \text{if} \quad \begin{cases} m < 2 \\ m = 2,\ k \geq 1 \\ m = 2, |w-z| > \mu(z,\frac{1}{\tau}) \end{cases}$$

If $m = 2$ and $|w - z| \leq \mu(z, \frac{1}{\tau})$,

$$|K_\tau(z,w)| \leq C \log\left(\frac{2\mu(z,\frac{1}{\tau})}{|w-z|}\right).$$

Also, if $\varphi \in C_c^\infty(D(z_0,\delta))$, we have the cancellation condition:

$$\sup_{z\in\mathbb{C}} \left| \int_{\mathbb{C}} (Y_{\tau p})^J K_\tau(z,w)\varphi(w)\,dw \right| \leq C_{\ell,n,k} \frac{\delta^{m-\ell}}{\tau^k \Lambda(z,\delta)^k} \sup_{w\in\mathbb{C}} \sum_{|I|\leq N_\ell} \delta^{|I|} \left|(X_{\tau p})^I \varphi(w)\right|.$$

$\Lambda(z,\delta)$ and $\mu(z,\delta)$ are geometric objects from the Carnot-Carathéodory geometry developed by Nagel, Stein, and Wainger [NSW85, Nag86]. The functions also arise in studying magnetic Schrödinger operators with electric potentials [She96, She99, Kur00].

The size and cancellation conditions form the heart of the definition. They allow us to prove:

Theorem 1.1. *If T_τ is an OPF operator of order 0, then T_τ, T_τ^* are bounded operators from $L^q(\mathbb{C})$ to $L^q(\mathbb{C})$, $1 < q < \infty$, with a constant independent of τ but depending on q.*

Also, the classes of OPF operators fufill the promise of being an analog to NIS operators. They are actually a tool with which NIS operators can be studied. We have the theorem:

Theorem 1.2. *Given a subharmonic, nonharmonic polynomial $p : \mathbb{C} \to \mathbb{R}$, there is a one-to-one correspondence between OPF operators of order $m \leq 2$ with respect to p and NIS operators of order $m \leq 2$ on the polynomial model domain $M_p = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 = p(z_1)\}$. The correspondence is given by a partial Fourier transform in $\operatorname{Re} z_2$.*

Now that we have introduced OPF operators and discussed two results about the families of operators, we are ready to analyze $\Box_{\tau p}$ and the heat operator $e^{-s\Box_{\tau p}}$. Studying $\Box_{\tau p} = -\bar{Z}_{\tau p} Z_{\tau p}$ instead of $\bar{L}L$ has two advantages. The first is that $\bar{Z}_{\tau p} Z_{\tau p}$ is elliptic, and the second is that we can express $2\Box_{\tau p} = \frac{1}{2}(-i\nabla - a)^2 + V$, a Schrödinger operator with magnetic potential a and electric potential V. These facts allow us to use a wealth of results unavailable in analyzing $\bar{L}L$.

Our analysis comes in two steps. First, we show that $e^{-s\Box_{\tau p}}$ is an integral operator with kernel $H_{\tau p}(s, z, w)$ that is smooth away from $\{(s, z, w) : z = w \text{ and } s = 0\}$. To do this, we use the ideas of [NS01] to develop properties of OPF operators. One essential result establishes the commutativity of $\bar{Z}_{\tau p}$ and $Z_{\tau p}$ with the class of order m families. From there, still following [NS01], we use the spectral theorem and L^2-methods to prove smoothness of $H_{\tau p}(s, z, w)$. Being able to work with OPF operators is critical in the proof because of the ability to change from arbitrary products of $\bar{Z}_{\tau p}$ and $Z_{\tau p}$ to powers of $\Box_{\tau p}$ composed with order 0 operators. Similarly to NIS operators, the spectral theorem gives the estimate $\|\Box_{\tau p}^j e^{-s\Box_{\tau p}}[\varphi]\|_{L^2(\mathbb{C})} \lesssim s^{-j}\|\varphi\|_{L^2(\mathbb{C})}$, and being able to write $X^\alpha = A_\tau \Box_{\tau p}^{|\alpha|/2}$ where A_τ is a well controlled OPF operator means we have the estimate

$$\|X^\alpha e^{-s\Box_{\tau p}}[\varphi]\|_{L^2(\mathbb{C})} \lesssim s^{-j}\|\varphi\|_{L^2(\mathbb{C})}. \tag{1.7}$$

The idea of the proof is to interpret the spectral theorem on the kernel side, that is, understand the implications of the spectral theorem on $H_{\tau p}(s, z, w)$. From (1.7), we see that $X^\alpha e^{-s\Box_{\tau p}}[\varphi] \in L^2(\mathbb{C})$ for all α. This allows us to prove that for a fixed s, $H_{\tau p}(s, z, w)$ and its derivatives are also in L^2. A Sobolev embedding theorem then implies that $H_{\tau p}$ is C^∞.

The second step of our analysis is to prove pointwise estimates on $H_{\tau p}(s, z, w)$ and its derivatives. We show:

Theorem 1.3. *Let $n \geq 0$ and Y^α be a product of $|\alpha|$ operators $Y = \bar{Z}_{\tau p}$ or $Z_{\tau p}$ if acting in z and $\overline{(Z_{\tau p})}$ or $\overline{(\bar{Z}_{\tau p})}$ if acting in w. There exists constants $c_1, c_2, c_3 > 0$ so that if $\tau > 0$,*

$$\left|\frac{\partial^n}{\partial s^n} Y^\alpha H_{\tau p}(s, z, w)\right| \leq c_1 \frac{1}{s^{n+\frac{1}{2}|\alpha|+1}} e^{-c_2\frac{|z-w|^2}{s}} e^{-c_3\frac{s}{\mu(z,\frac{1}{\tau})^2}}.$$

The estimates are shown in two stages. In the first stage, if $z = x_1 + ix_2$, similarly to Berndtsson [Ber96] we write $2\Box_{\tau p} = \frac{1}{2}(-i\nabla - a)^2 + V$ where

$$a(z) = \tau \begin{pmatrix} -\dfrac{\partial p}{\partial x_2} \\ \dfrac{\partial p}{\partial x_1} \end{pmatrix} \qquad \text{and} \qquad V(z) = \frac{1}{2}\tau \triangle p(z).$$

We use the Feynman-Kac-Itô formula [Sim79] to show Gaussian decay for $H_{\tau p}(s, z, w)$. To finish the estimate on $|H_{\tau p}(s, z, w)|$, we scale to normalize the polynomial p, prove an L^2-"energy" inequality, and finish the argument with Grönwall's inequality and $e^{-s\Box_{\tau p}}$

semigroup properties. We show the estimate on a compact class of polynomials of degree $2n$ and scale the result to obtain the estimates in the general case.

The second stage of the estimation is to prove pointwise bounds on the derivatives $\frac{\partial^n}{\partial s^n}U^\alpha H_{\tau p}(s,z,w)$. The idea is to prove a local L^2-bound for $\frac{\partial^n}{\partial s^n}U^\alpha H_{\tau p}(s,z,w)$ and its derivatives and pass to a local L^∞-bound using either a Sobolev embedding-type result, Theorem 3.12 or the subsolution estimation from Kurata [Kur00], Lemma A.1. The local L^2-bound uses an integration by parts argument, and the ability to change arbitrary derivatives into derivatives controlled by the spectral theorem is critical.

1.6 Conclusion and Future Directions

Kurata studied heat kernels in $\mathbb{R}^n$ for Schrödinger operators of the form $L = (-i\nabla - a)^2 + V$ where $a \in C^1$ and $V \in L^q_{loc}(\mathbb{R}^n)$, $V \geq 0$. His conditions on a and V are more general than what we consider, and he can only prove continuity of the heat kernel. He shows the bound $|H_{\tau p}(s,z,w)| \leq \frac{C}{s} e^{-c_2\frac{|z-w|^2}{s}} e^{-c_3\left(\frac{s}{\mu(z,\frac{1}{\tau})^2}\right)^{1/2m}}$, a weaker result than what we have shown. The lack of structure of L prevents the technique of Theorem 1.3 from applying to Kurata's more general operators.

By integrating in s, the pointwise estimates on $H_{\tau p}(s,z,w)$ allow us to recover estimates on the fundamental solution of $\Box_{\tau p}$ and compare our work to Christ [Chr91a]. If $G_{\tau p}(z,w)$ is the fundamental solution to $\Box_{\tau p}$, we show the decay:

Corollary 1.4. *Let $G_{\tau p}(z,w)$ be the fundamental solution for $\Box_{\tau p}^{-1}$. There exists constants $C_1, C_2 > 0$ so that if $\tau > 0$,*

$$|G_{\tau p}(z,w)| \leq C_1 \begin{cases} \log\left(\frac{2\mu(z,\frac{1}{\tau})}{|z-w|}\right) & \mu(z,\frac{1}{\tau}) \geq |z-w| \\ e^{-C_2\frac{|z-w|}{\mu(z,\frac{1}{\tau})}} & \mu(z,\frac{1}{\tau}) \leq |z-w| \end{cases}$$

Near the diagonal, our estimates agree with Christ, but far from the diagonal, he shows the decay $e^{-c\rho_{\tau p}(z,w)}$ where $\rho_{\tau p}$ is a metric on $\mathbb{C}$ which reflects the geometry of the measure $\tau\triangle p$ and $\rho_{\tau p}(z,w) \geq C\frac{|w-z|}{\mu(z,\frac{1}{\tau})}$. The advantage to our estimates, however, is that we can compute them easily since μ and Λ are calculated directly from the coefficients of τp. $\rho(z,w)$ is difficult to calculate.

In $\mathbb{R}^n$, $n \geq 3$, Shen ('99) obtains estimates for the decay of the fundamental solution of $-\triangle + V$, V is a nonnegative Radon measure. Interestly, his estimates are sharp even though they are higher dimensional versions of Christ's estimates which are not sharp. This signifies there is additional structure in the special relationship between a and V in the magnetic Schrodinger operator $\Box_{\tau p}$ which has not been exploited.

Estimates on $H_{\tau p}(s,z,w)$ have many applications that we would like to explore. The first step in this direction is to obtain pointwise estimates for $H_{\tau p}(s,z,w)$ and its derivatives when $\tau < 0$. This is equivalent to finding estimates on $\tilde{H}_{\tau p}(s,z,w)$ when $\tau > 0$. Unfortunately, the techniques from parabolic operator theory and quantum mechanics do not seem to work. The difficulty lies in the fact that when we write $2\widetilde{\Box}_{\tau p} = \frac{1}{2}(-i\nabla - \tilde{a})^2 + \tilde{V}$, $\tilde{V} \leq 0$. Writing $\widetilde{\Box}_{\tau p}$ as a parabolic operator, this means the 0^{th} order term may not be positive. I am in the process of developing methods which exploit the structure of $\Box_{\tau p}$ to overcome these difficulties. Once that is accomplished, we plan to use our results to prove exponential decay for the heat kernel of [NS01], an improvement over the rapid decay shown by Nagel and Stein. We also hope to use the OPF operator and heat kernel results to build on the work of [NS03] by proving pointwise estimates on the heat kernel on decoupled domains in $\mathbb{C}^n$, i.e. domains of the form $M = \{(z_1, \ldots, z_n) : \operatorname{Im} z_n = \sum_{j=1}^{n-1} p_j(z_j)\}$ where p_j are nonharmonic, subharmonic polynomials.

Chapter 2

One-Parameter Families of Operators on $\mathbb{C}$

2.1 Notation and Definitions

2.1.1 Notation

Let p be a subharmonic, nonharmonic polynomial of degree $2n$. On $\mathbb{C} \times \mathbb{R}$, we define the operators

$$\bar{L}_z = \frac{\partial}{\partial \bar{z}} - i\frac{\partial p}{\partial \bar{z}}\frac{\partial}{\partial t} \qquad \bar{\mathcal{L}}_w = \frac{\partial}{\partial \bar{w}} + i\frac{\partial p}{\partial \bar{w}}\frac{\partial}{\partial t}$$
$$L_z = \frac{\partial}{\partial z} + i\frac{\partial p}{\partial z}\frac{\partial}{\partial t} \qquad \mathcal{L}_w = \frac{\partial}{\partial w} - i\frac{\partial p}{\partial w}\frac{\partial}{\partial t}.$$

If M_p is a polynomial model domain in $\mathbb{C}^2$ given by $M_p = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 = p(z_1)\}$, then $M_p \cong \mathbb{C} \times \mathbb{R}$ and $\bar{\partial}_b$ (defined on M) becomes the operator $\bar{L}_z$ on $\mathbb{C} \times \mathbb{R}$. It follows that $-L_z$ is the Hilbert space adjoint to $\bar{L}_z$ in $L^2(\mathbb{C} \times \mathbb{R})$. The translation invariance in t causes many operators of interest to have a convolution structure in t. A consequence is that if we have a function $\tilde{f}\big((z,t),(w,s)\big) = f(z, w, t-s)$, we may study $f(z, w, t)$. By the chain rule, $\bar{\mathcal{L}}_w$ and $\mathcal{L}_w$ are the versions of $\bar{L}_z$ and L_z in the w-variable.

We take the partial Fourier transform in t of the operators $\bar{L}_z$, L_z, $\bar{\mathcal{L}}_w$, $\mathcal{L}_w$. We let the τ be the transform variable of t, and we define the operators

$$Z_{\tau p,z} = \frac{\partial}{\partial z} - \tau\frac{\partial p}{\partial z} \qquad W_{\tau p,w} = \frac{\partial}{\partial w} + \tau\frac{\partial p}{\partial w}$$
$$\bar{Z}_{\tau p,z} = \frac{\partial}{\partial \bar{z}} + \tau\frac{\partial p}{\partial \bar{z}} \qquad \overline{W}_{\tau p,w} = \frac{\partial}{\partial \bar{w}} - \tau\frac{\partial p}{\partial \bar{w}}.$$

We think of τ as fixed and the operators $\bar{Z}_{\tau p,z}$, $Z_{\tau p,z}$, $\overline{W}_{\tau p,w}$, and $W_{\tau p,w}$ as acting on functions defined on $\mathbb{C}$. We will omit the variables z and w from subscripts when the application is unambiguous. Observe that $\overline{(Z_{\tau p})} = \overline{W}_{\tau p}$ and $\overline{(\bar{Z}_{\tau p})} = W_{\tau p}$. We let X_1 and X_2 denote the "real" and "imaginary" parts of Z, that is,

$$X_1 = Z_{\tau p} + \bar{Z}_{\tau p} = \frac{\partial}{\partial x_1} + i\tau\frac{\partial p}{\partial x_2}$$
$$X_2 = i(Z_{\tau p} - \bar{Z}_{\tau p}) = \frac{\partial}{\partial x_2} - i\tau\frac{\partial p}{\partial x_1}.$$

Analogously to X_1 and X_2, define

$$U_1 = W_{\tau p} + \overline{W}_{\tau p} = \frac{\partial}{\partial x_1} - i\tau\frac{\partial p}{\partial x_2}$$
$$U_2 = i(W_{\tau p} - \overline{W}_{\tau p}) = \frac{\partial}{\partial x_2} + i\tau\frac{\partial p}{\partial x_1}.$$

We need to establish notation for adjoints. If T is an operator (either bounded or closed and densely defined) on a Hilbert space with inner product $\big(\cdot\,,\cdot\big)$, let T^* be the Hilbert space adjoint of T. This means that if $f \in \operatorname{Dom} T$ and $g \in \operatorname{Dom} T^*$, then $\Big(Tf, g\Big) = \Big(f, T^*g\Big)$. The Hilbert spaces that arise in this thesis are $L^2(\mathbb{C})$, $L^2(\mathbb{C}\times\mathbb{R})$, and $L^2(\mathbb{R}\times\mathbb{C})$. If $U = \mathbb{C}$, $\mathbb{C}\times\mathbb{R}$, or $\mathbb{R}\times\mathbb{C}$ and T is an operator acting on $C^\infty_c(U)$ or $\mathcal{S}(U) = \{\varphi \in C^\infty(U) : \varphi \text{ has rapid decay}\}$, then we denote $T^{\#}$ as the adjoint in the sense of distributions. This means is K is a distribution or a Schwartz distribution, then $\langle T^{\#}K, \varphi\rangle = \langle K, T\varphi\rangle$. Note that if T is not $\mathbb{R}$-valued, $T^* \neq T^{\#}$. It follows easily that

$$\bar{Z}^{\#}_{\tau p} = -\overline{W}_{\tau p} \qquad \text{and} \qquad Z^{\#}_{\tau p} = -W_{\tau p}.$$

Notation for Carnot-Carathéodory geometry.

The metric and corresponding balls from the Carnot-Carathéodory geometry on polynomial model domains play an important role in this work. We need the following functions in order to describe the metric ρ. Let

$$\Lambda(z,\delta) = \sum_{j,k\geq 1} \left|a^z_{jk}\right| \delta^{j+k} \tag{2.1}$$

where

$$a^z_{jk} = \frac{1}{j!k!}\frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(z). \tag{2.2}$$

Define

$$\mu(z,\delta) = \inf_{j,k\geq 1} \frac{|\delta|^{1/j+k}}{|a^z_{jk}|^{1/j+k}}, \tag{2.3}$$

and we see that $\mu(z,\delta)$ is an approximate inverse to $\Lambda(z,\delta)$. This means that if $\delta > 0$,

$$\mu\big(z,\Lambda(z,\delta)\big) \sim \delta \text{ and } \Lambda\big(z,\mu(z,\delta)\big) \sim \delta.$$

We use the notation $a \lesssim b$ if $a \leq Cb$ where C is a constant that may depend on the dimension 2 and the degree of p. We say that $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

We also denote the "twist" at w, centered as z by

$$\begin{aligned} T(w,z) &= -2\operatorname{Im}\left(\sum_{j\geq 1}\frac{1}{j!}\frac{\partial^j p}{\partial z^j}(z)(w-z)^j\right) \\ &= i\left(\sum_{j\geq 1}\frac{1}{j!}\frac{\partial^j p}{\partial z^j}(z)(w-z)^j - \sum_{j\geq 1}\frac{1}{j!}\frac{\partial^j p}{\partial \bar{z}^j}(z)\overline{(w-z)}^j\right). \end{aligned} \tag{2.4}$$

Given $(z,t),(w,s) \in \mathbb{C}\times\mathbb{R}$, the nonisotropic distance

$$\rho\big((z,t),(w,s)\big) = |z-w| + \mu\big(z, t-s+T(w,z)\big).$$

Since $\rho\big((z,t),(w,s)\big)$ is a function of z, w, and $t-s$, we define a new function

$$d_{NI}(z,w,t) = |z-w| + \mu\big(z, t+T(w,z)\big). \tag{2.5}$$

We will see that $d_{NI}(z,w,t)$ is essentially symmetric in (z,w). The nonisotropic ball

$$B_{NI}\big((z,t),\delta\big) = \{(w,s) : d_{NI}(z,w,t-s) < \delta\}.$$

We also define a volume function

$$V_{NI}\big((z,t),(w,s)\big) = \Big|B_{NI}\big((z,t),d_{NI}(z,w,t-s)\big)\Big| = d_{NI}(z,w,t-s)^2\Lambda\big(z,d_{NI}(z,w,t-s)\big).$$

Properties of $T(w,z)$.

Proposition 2.1.

$$T(w,z) = -T(z,w)$$

Proof. Since $p(z) = \sum_{j,k} \frac{1}{j!k!}\frac{\partial^{j+k}p}{\partial z^j\partial\bar{z}^k}(w)(z-w)^j\overline{(z-w)}^k$, we have

$$\frac{\partial^\ell p}{\partial z^\ell}(z) = \sum_{\substack{j\geq\ell\\k\geq 0}} \frac{j(j-1)\cdots(j-(\ell-1))}{j!k!}\frac{\partial^{j+k}p}{\partial z^j\partial\bar{z}^k}(w)(z-w)^{j-\ell}\overline{(z-w)}^k$$
$$\sum_{\substack{j\geq\ell\\k\geq 0}} \frac{j!}{(j-\ell)!}\frac{1}{j!k!}.\frac{\partial^{j+k}p}{\partial z^j\partial\bar{z}^k}(w)(z-w)^{j-\ell}\overline{(z-w)}^k$$

Since p is $\mathbb{R}$-valued, the twist (equation (2.4)) $T(w,z) = -2\,\mathrm{Im}\Big(\sum_{\ell\geq 0}\frac{1}{\ell!}\frac{\partial^\ell p}{\partial z^\ell}(z)(w-z)^\ell\Big)$, so

$$\sum_{\ell\geq 0}\frac{1}{\ell!}\frac{\partial^\ell p}{\partial z^\ell}(z)(w-z)^\ell$$
$$= \sum_{\ell\geq 0}\frac{1}{\ell!}\left(\sum_{\substack{j\geq\ell\\k\geq 0}} \frac{j!}{(j-\ell)!}\frac{1}{j!k!}.\frac{\partial^{j+k}p}{\partial z^j\partial\bar{z}^k}(w)(z-w)^{j-\ell}\overline{(z-w)}^k\right)(w-z)^\ell$$
$$= \sum_{\ell\geq 0}\sum_{\substack{j\geq\ell\\k\geq 0}} \frac{j!}{\ell!(j-\ell)!}\frac{(-1)^\ell}{j!k!}\frac{\partial^{j+k}p}{\partial z^j\partial\bar{z}^k}(w)(z-w)^j\overline{(z-w)}^k$$

$$= \sum_{\substack{j\geq 0\\ k\geq 0}} \left(\sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^{\ell} \right) \frac{1}{j!k!} \frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(w)(z-w)^j \overline{(z-w)}^k$$

$$= \sum_{k\geq 0} \frac{1}{k!} \frac{\partial^k p}{\partial \bar{z}^k}(w)\overline{(z-w)}^k = \overline{\sum_{j\geq 0} \frac{1}{j!} \frac{\partial^j p}{\partial z^j}(w)(z-w)^j}.$$

The second to last line uses the identity $\sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^{\ell} = \delta_0(j)$. The result follows easily. □

Corollary 2.2.

$$d_{NI}(z,w,t) \sim d_{NI}(w,z,t).$$

Proof. This is a well known fact ([NSW85, Nag86]), but we are in a situation where the computations can be explicit. We sketch a proof. If $r = |t + T(w,z)|$, it follows from from Proposition 2.1 that it is enough to show that

$$|z-w| + \mu(z,r) \sim |z-w| + \mu(w,r).$$

If $\mu(z,r) < |z-w|$ and $\mu(w,r) < |z-w|$, there is nothing to prove, so (without loss of generality) assume that $\mu(z,r) > |z-w|$. By expanding $p(z)$ around w and $p(w)$ around z, it can be shown that $\Lambda(z,\delta) \sim \Lambda(w,\delta)$ if $\delta > |w-z|$. Thus, we see

$$\Lambda\big(w, \mu(z,r)\big) \sim \Lambda\big(z, \mu(z,r)\big) \sim r,$$

and it follows that $\mu(z,r) \sim \mu(w,r)$. □

The next proposition contains two useful, though simple, computations.

Proposition 2.3.

$$\frac{\partial T}{\partial z}(w,z) = -i\frac{\partial p}{\partial z}(z) - i\sum_{j\geq 1} \frac{1}{j!} \frac{\partial^{j+1} p}{\partial z \partial \bar{z}^j}(z)\overline{(w-z)}^j$$

and

$$\frac{\partial T}{\partial \bar{z}}(w,z) = i\frac{\partial p}{\partial \bar{z}}(z) + i\sum_{j\geq 1}\frac{1}{j!}\frac{\partial^{j+1}p}{\partial z^j \partial \bar{z}}(z)(w-z)^j.$$

Proof. The proof is a short computation.

$$\begin{aligned}\frac{\partial T}{\partial z}(w,z) &= i\Bigg(\sum_{j=1}^{2n-1}\frac{1}{j!}\frac{\partial^{j+1}p}{\partial z^{j+1}}(z)(w-z)^j - \sum_{j=1}^{2n}\frac{1}{(j-1)!}\frac{\partial^j p}{\partial z^j}(z)(w-z)^{j-1} \\ &\qquad - \sum_{j=1}^{2n-1}\frac{1}{j!}\frac{\partial^{j+1}p}{\partial z\partial \bar{z}^j}(z)\overline{(w-z)}^j\Bigg) \\ &= -i\frac{\partial p}{\partial z}(z) - i\sum_{j\geq 1}\frac{1}{j!}\frac{\partial^{j+1}p}{\partial z\partial \bar{z}^j}(z)\overline{(w-z)}^j,\end{aligned}$$

noting that the first sum cancels all but the first term of the second sum. Since T is $\mathbb{R}$-valued, $\frac{\partial T}{\partial \bar{z}}(w,z) = \overline{\frac{\partial T}{\partial z}(w,z)}$ which gives the result for the second sum. □

A useful consequence of these calculations is

Proposition 2.4. *Let $\mathcal{Y}^J$ be a product of $|J|$ operators of the form $\mathcal{Y}^j = L_z, \bar{L}_z, \mathcal{L}_w, \bar{\mathcal{L}}_w$. Then*

$$|\mathcal{Y}^J\big(t + T(w,z)\big)| \leq C_{|J|}\frac{\Lambda(z, d_{NI}(z,w,t))}{d_{NI}(z,w,t)^{|J|}}.$$

Before we prove the Proposition 2.4, we note that the result would be false if we replaced $t+T(w,z)$ with t or $T(w,z)$. Without both terms, there would be uncontrolled derivatives of p remaining after applying $\mathcal{Y}^j$.

Proof. We have $L_z\big(t+T(w,z)\big) = \frac{\partial T}{\partial z}(w,z) + i\frac{\partial p}{\partial z}(z) = -i\sum_{j\geq 1}\frac{1}{j!}\frac{\partial^{j+1}p(z)}{\partial z\partial \bar{z}^j}\overline{(w-z)}^j$. Similarly, $\bar{L}_z\big(t+T(w,z)\big) = i\sum_{j\geq 1}\frac{1}{j!}\frac{\partial^{j+1}p(z)}{\partial z^j\partial \bar{z}}(w-z)^j$. Analogous equalities (with z and w interchanged and the sign switched) hold for $\mathcal{L}_w\big(t+T(w,z)\big)$ and $\bar{\mathcal{L}}_w\big(t+T(w,z)\big)$ since

$$\mathcal{L}_w\big(t+T(w,z)\big) = \left(\frac{\partial}{\partial w} - i\frac{\partial p}{\partial w}\frac{\partial}{\partial t}\right)(t - T(z,w)) = -i\frac{\partial p}{\partial w}(w) - \frac{\partial T}{\partial w}(z,w)$$
$$= -\left(i\frac{\partial p}{\partial w}(w) + \frac{\partial T}{\partial w}(z,w)\right) = -\left(\frac{\partial}{\partial w} + i\frac{\partial p}{\partial w}\frac{\partial}{\partial t}\right)(t + T(z,w))$$
$$= -L_w(t + T(z,w))$$

and $\bar{\mathcal{L}}_w\big(t+T(w,z)\big) = -\bar{L}_w(t+T(z,w))$ But

$$\left|\sum_{j\geq 1}\frac{1}{j!}\frac{\partial^{j+1}p(z)}{\partial z^j \partial \bar{z}}(w-z)^j\right| \leq c_1 \frac{\Lambda(z, d_{NI}(z,w,t))}{d_{NI}(z,w,t)}.$$

Higher order derivatives are easier. As we just showed, the result of applying $\mathcal{Y}^1$ to $t + T(w,z)$ leaves a polynomial that is a sum of derivatives of $\triangle p$ (and hence well controlled). There are no t terms remaining, so if $j \geq 2$, applying $\mathcal{Y}^j$ is a matter of applying one of: $\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial w}$. Hence, the computation is simpler, and it can be done naively, i.e. there is no need to find any cancelling terms (which in general are not present anyway). $\square$

Finally, let

$$M_{\tau p} = e^{i\tau T(w,z)}\frac{\partial}{\partial \tau}e^{-i\tau T(w,z)}, \text{ and } \mathcal{M} = -i\big(t + T(w,z)\big).$$

2.1.2 Definition of OPF Operators

Let p be a subharmonic, nonharmonic polynomial. We say that T_τ is a *one-parameter family (OPF) of operators* of order m with respect to the polynomial p if the following conditions hold:

(a) There is a function $K_\tau \in C^\infty\Big(\big((\mathbb{C}\times\mathbb{C})\setminus\{z=w\}\big)\times(\mathbb{R}\setminus\{0\})\Big)$ so that for fixed τ, K_τ is a distributional kernel, i.e. if $\varphi, \psi \in C^\infty_c(\mathbb{C})$ and $\operatorname{supp}\varphi \cap \operatorname{supp}\psi = \emptyset$, then

$T_\tau[\varphi] \in (C_c^\infty)'(\mathbb{C})$ and

$$\langle T_\tau[\varphi](\cdot), \psi\rangle_{\mathbb{C}} = \iint_{\mathbb{C}\times\mathbb{C}} K_\tau(z,w)\varphi(w)\psi(z)\, dwdz.$$

(b) There exists a family of functions $K_{\tau,\epsilon}(z,w) \in C^\infty(\mathbb{C}\times\mathbb{C}\times\mathbb{R})$ so that if $\varphi \in C_c^\infty(\mathbb{C}\times\mathbb{R})$,

$$K_{\tau,\epsilon}[\varphi]_{\mathbb{C}\times\mathbb{R}}(z,\tau) = \int_{\mathbb{C}\times\mathbb{R}} \varphi(w,\tau)K_{\tau,\epsilon}(z,w)\, dwd\tau$$

and $\lim_{\epsilon\to 0} K_{\tau,\epsilon}[\varphi]_{\mathbb{C}\times\mathbb{R}}(z) = K_\tau[\varphi]_{\mathbb{C}\times\mathbb{R}}(z)$ in $(C_c^\infty)'(\mathbb{C}\times\mathbb{R})$.

All of the additional conditions are assumed to apply the kernels $K_{\tau,\epsilon}(z,w)$ uniformly in ϵ.

(c) Size Estimates. If $Y_{\tau p}^J$ is a product of $|J|$ operators of the form $Y_{\tau p}^j = Z_{\tau p,z}$, $\bar{Z}_{\tau p,z}$, $W_{\tau p,w}$, $\overline{W}_{\tau p,w}$, or $M_{\tau p}$ where $|J| = \ell + n$ and $n = \#\{j : Y_{\tau p}^j = M_{\tau p}\}$, there exists a constant $C_{\ell,n,k}$ so that for any k,

$$\left|Y_{\tau p}^J K_{\tau,\epsilon}(z,w)\right| \leq C_{\ell,n,k}\frac{|z-w|^{m-2-\ell}}{\tau^{n+k}\Lambda(z,|w-z|)^k} \quad \text{if} \quad \begin{cases} m<2 \\ m=2,\ k\geq 1 \\ m=2, |w-z|>\mu(z,\frac{1}{\tau}) \end{cases} \tag{2.6}$$

Also, if $m=2$ and $|w-z| \leq \mu(z,\frac{1}{\tau})$, then

$$\left|M_{\tau p}^n K_{\tau,\epsilon}(z,w)\right| \leq C_n \begin{cases} \log\left(\frac{2\mu(z,\frac{1}{\tau})}{|w-z|}\right) & n=0 \\ |\tau|^{-n} & n\geq 1 \end{cases} \tag{2.7}$$

(d) Cancellation in w. If $Y_{\tau p}^J$ is a product of $|J|$ operators of the form $Y_{\tau p}^j = Z_{\tau p}$, $\bar{Z}_{\tau p}$, or $M_{\tau p}$ where $|J| = \ell + n$ and $n = \#\{j : Y_{\tau p}^j = M_{\tau p}\}$, there exists a constant $C_{\ell,n,k}$

and N_ℓ so that so that for $\varphi \in C_c^\infty(D(z_0,\delta))$ and any k,

$$\sup_{z\in\mathbb{C}} \left| \int_{\mathbb{C}} Y_{\tau p}^J K_{\tau,\epsilon}(z,w)\varphi(w)\,dw \right| \leq C_{\ell,n,k} \frac{\delta^{m-\ell}}{\tau^{k+n}\Lambda(z,\delta)^k} \sup_{w\in\mathbb{C}} \sum_{|I|\leq N_\ell} \delta^{|I|} \left| X_{\tau p}^I \varphi(w) \right| \quad (2.8)$$

where $X_{\tau p}^I$ is composed solely of $Z_{\tau p}$ and $\bar{Z}_{\tau p}$.

(e) Cancellation in τ. If $X_{\tau p}^J$ is a product of $|J|$ operators of the form $X_{\tau p}^j = Z_{\tau p,z},\ \bar{Z}_{\tau p,z}$ or $W_{\tau p,w},\ \overline{W}_{\tau p,w}$ and $|J| = n$, there exists a constant C_n so that

$$\int_{\mathbb{R}} X_{\tau p}^J \left(e^{i\tau t} K_{\tau,\epsilon}(z,w) \right)\,d\tau \leq C_n \frac{\mu(z,t+T(w,z))^{m-n}}{\mu(z,t+T(w,z))^2 |t+T(w,z)|}. \quad (2.9)$$

(f) Adjoint. Properties (a)-(e) also hold for the adjoint operator T_τ^* whose distribution kernel is given by $\overline{K_{\tau,\epsilon}(w,z)}$

Note that for the τ-cancellation condition (2.9), we do not need to consider the case $X_{\tau p}^j = \frac{\partial}{\partial\tau}$ since $\int_{\mathbb{R}} \frac{\partial}{\partial\tau}\left(e^{i\tau(t+T(w,z))} K_{\tau,\epsilon}(z,w)\right) d\tau = 0$.

In size condition (c) and cancellation condition (d), the $\tau^k\Lambda(z,|z-w|)^k$ and $\tau^k\Lambda(z,\delta)^k$ terms are rapid decay terms. If OPF operators are to be partial Fourier transforms of NIS operators on polynomial model domains, rapid decay should not be surprising; it is consequence of being able to integrate parts from the Fourier transform formula. This will be seen explicitly in Lemma 2.11. Ignoring the rapid decay terms, the size and cancellation conditions of OPF operators are familiar. An order 2 OPF operator should "invert" two derivatives, like the Newtonian potential. In $\mathbb{R}^2$, the Newtonian potential has a logarithmic blowup on the diagonal, just like an order 2 OPF operator. For an order 0 OPF operator, the blowup on the diagonal is the same as a Calderòn-Zygmund kernel, and the decay of $K_\tau(0,z)$ is $|z|^{-2}$, the same as a Calderòn-Zygmund kernel. For the cancellation conditions, if φ is "normalized" appropriately, the cancellation condition

(2.8) simplifies to

$$\|Y_{\tau p}^{J} T_\tau[\varphi]\|_{L^\infty(\mathbb{C})} \lesssim \delta^j.$$

This is reminicent of cancellation of a Calderòn-Zygmund operator or an NIS operator.

2.2 L^q boundedness of order 0 operators

We are now ready to prove Theorem 1.1.

Theorem 1.1. *If T_τ is an OPF operator of order 0, then T_τ, T_τ^* are bounded operators from $L^q(\mathbb{C})$ to $L^q(\mathbb{C})$, $1 < q < \infty$, with a constant independent of τ but depending on q.*

The idea is to show that $e^{-i\tau T(w,z)} K_{\tau,\epsilon}$ satisfies the bounds of a Calderon-Zygmund kernel and the operator S_τ with kernel $e^{-i\tau T(w,z)} K_{\tau,\epsilon}$ is restrictly bounded. These two facts will allow us to use the $T(1)$ theorem to prove that S_τ is bounded on $L^2(\mathbb{C})$, and a result by Ricci and Stein [RS87] applies and proves boundedness of T_τ on $L^q(\mathbb{C})$.

We need the follow two lemmas.

Lemma 2.5. *Let T_τ be a family of operators of order $m \leq 2$ with a family of kernel approximating functions $K_{\tau,\epsilon}$. Fixing τ, $K_{\tau,\epsilon}(z,w)$ satisfies:*

(a)

$$\left|\nabla_{z,w}\left(e^{-i\tau T(w,z)} K_{\tau,\epsilon}(z,w)\right)\right| \leq C_k \frac{|w-z|^{m-3}}{|\tau|^k \Lambda(z,|w-z|)^k} \tag{2.10}$$

(b) If $2|w-w'| \leq |w-z|$, then

$$\left|e^{-i\tau T(w,z)} K_{\tau,\epsilon}(z,w) - e^{-i\tau T(w',z)} K_{\tau,\epsilon}(z,w')\right| \leq C_k \frac{|w-w'|}{|w-z|^{3-m}|\tau|^k \Lambda(z,|w-z|)^k} \tag{2.11}$$

(c) If $2|z-z'| \le |w-z|$, *then*

$$\left|e^{-i\tau T(w,z)}K_{\tau,\epsilon}(z,w) - e^{-i\tau T(w,z')}K_{\tau,\epsilon}(z',w)\right| \le C_k \frac{|z-z'|}{|w-z|^{3-m}|\tau|^k\Lambda(z,|w-z|)^k} \tag{2.12}$$

Proof. It is immediate that the Mean Value Theorem shows (2.10) implies (2.11) and (2.12). To prove (2.10), we use Proposition 2.3 and compute:

$$\begin{aligned}
&e^{i\tau T(w,z)}\frac{\partial}{\partial z}\left(e^{-i\tau T(w,z)}K_{\tau,\epsilon}(z,w)\right)\\
&\qquad = -i\tau\frac{\partial T}{\partial z}(w,z)K_{\tau,\epsilon}(z,w) + \frac{\partial K_{\tau,\epsilon}}{\partial z}(z,w)\\
&\qquad = \frac{\partial K_{\tau,\epsilon}}{\partial z}(z,w) - \tau\frac{\partial p}{\partial z}(z)K_{\tau,\epsilon}(z,w) - \tau\sum_{j\ge 1}\frac{1}{j!}\frac{\partial^{j+1}p}{\partial z\partial\bar{z}^j}(z)\overline{(w-z)}^j K_{\tau,\epsilon}(z,w).
\end{aligned}$$

Using the size estimate (2.6),

$$\begin{aligned}
\left|\frac{\partial}{\partial z}\left(e^{-i\tau T(w,z)}K_{\tau,\epsilon}(z,w)\right)\right| &\le Z_{\tau p}K_{\tau,\epsilon}(z,w) + \frac{\tau\Lambda(z,|w-z|)}{|w-z|}K_{\tau,\epsilon}(z,w)\\
&\le C_k\frac{|w-z|^{m-3}}{|\tau|^k\Lambda(z,|w-z|)^k}.
\end{aligned}$$

A virtually identical calculation shows

$$\left|\frac{\partial}{\partial \bar{z}}\left(e^{-i\tau T(w,z)}K_{\tau,\epsilon}(z,w)\right)\right| \le C_k\frac{|w-z|^{m-3}}{|\tau|^k\Lambda(z,|w-z|)^k}$$

which shows $\left|\frac{\partial}{\partial \bar{z}}\left(e^{-i\tau T(w,z)}K_{\tau,\epsilon}(z,w)\right)\right|$ satisfies the bound in (2.10). The bounds for $\left|\frac{\partial}{\partial w}\left(e^{-i\tau T(w,z)}K_{\tau,\epsilon}(z,w)\right)\right|$ and $\left|\frac{\partial}{\partial \bar{w}}\left(e^{-i\tau T(w,z)}K_{\tau,\epsilon}(z,w)\right)\right|$ use a repetition of the calculations just performed and the identity $e^{-i\tau T(w,z)} = e^{i\tau T(z,w)}$ (which follows from Proposition 2.1). □

We now restrict ourselves to the case $m=0$. Given an family T_τ of order 0, define a related family of operators S_τ so that if $K_\tau(z,w)$ is the kernel of T_τ, the kernel of S_τ is given by $e^{-i\tau T(w,z)}K_\tau(z,w)$. We have the following:

Lemma 2.6. *S_τ and S_τ^* are restrictly bounded, i.e. if $\varphi \in C_c^\infty(D(0,1))$, $\|\varphi\|_{C_{N_0}} \leq 1$ (where N_0 is the constant from the cancellation condition (2.8)) and $\varphi^{R,z_0}(z) = \varphi(\frac{z-z_0}{R})$, then*

$$\|S_\tau(\varphi^{R,z_0})\|_{L^2} \leq AR, \qquad \|(S_\tau)^*(\varphi^{R,z_0})\|_{L^2} \leq AR$$

with the constant A independent of τ.

Proof. From the adjoint condition (f), it follows that we only have the prove the restricted boundedness of S_τ.

$$\begin{aligned}
\|S_{\tau,\epsilon}(\varphi^{R,z_0})\|_{L^2} &= \left(\int_{\mathbb{C}} \left| \int_{\mathbb{C}} e^{-i\tau T(w,z)} K_{\tau,\epsilon}(z,w)\varphi(\tfrac{w-z_0}{R})\, dw \right|^2 dz\right)^{\frac{1}{2}} \\
&\leq \left(\int_{|z-z_0|<2R} \left| \int_{\mathbb{C}} K_{\tau,\epsilon}(z,w) \left(e^{-i\tau T(w,z)}\varphi(\tfrac{w-z_0}{R})\right)\, dw \right|^2 dz\right)^{\frac{1}{2}} \\
&+ \left(\int_{|z-z_0|\geq 2R} \left| \int_{\mathbb{C}} e^{-i\tau T(w,z)} K_{\tau,\epsilon}(z,w)\varphi(\tfrac{w-z_0}{R})\, dw \right|^2 dz\right)^{\frac{1}{2}} \\
&= I + II.
\end{aligned}$$

We estimate I first. By the cancellation condition (2.8)

$$\begin{aligned}
&\left| \int_{\mathbb{C}} K_{\tau,\epsilon}(z,w) \left(e^{-i\tau T(w,z)}\varphi(\tfrac{w-z_0}{R})\right)\, dw \right| \\
&\qquad \leq C_{N_0} \frac{1}{\max\{1, |\tau|^{N_0}\Lambda(z,R)^{N_0}\}} \sup_{w\in\mathbb{C}} \sum_{|I|\leq N_0} R^{|I|} \left| Y_{\tau p}^I \left(e^{-i\tau T(w,z)}\varphi(\tfrac{w-z_0}{R})\right)\right|.
\end{aligned}$$

We claim $R^{|I|} \left| Y_{\tau p}^I \left(e^{i\tau T(z,w)}\varphi(\frac{w-z_0}{R})\right)\right| \leq C_{|I|} \max\{1, |\tau|^{|I|}\Lambda(z,R)^{|I|}\}$. To see this, we first do the case $Y_{\tau p}^I = Z_{\tau p,w}$. It follows from Proposition 2.1 and Proposition 2.3 that

$$\begin{aligned}
&Z_{\tau p,w}\left(e^{i\tau T(z,w)}\varphi(\tfrac{w-z_0}{R})\right) \\
&\qquad = \frac{e^{i\tau T(z,w)}}{R}\frac{\partial \varphi}{\partial w}(\tfrac{w-z_0}{R}) + \tau e^{i\tau T(z,w)} \sum_{j\geq 1} \frac{1}{j!}\frac{\partial^{j+1} p}{\partial w \partial \bar{w}^j}(w)\overline{(z-w)}^j \varphi(\tfrac{w-z_0}{R}).
\end{aligned}$$

Hence $\left|Z_{\tau p,w}\left(e^{i\tau T(z,w)}\varphi(\frac{w-z_0}{R})\right)\right| \leq \frac{C}{R}\left(1+\tau\Lambda(z,R)\right)$. Iterating this argument proves the claim. Thus, for $|z-z_0| \leq 2R$,

$$\left|\int_{\mathbb{C}} K_{\tau,\epsilon}(z,w)e^{-i\tau T(w,z)}\varphi(\tfrac{w-z_0}{R})\,dw\right| \leq C,$$

and

$$I \leq C\left(\int_{|z-z_0|<2R} dz\right)^{\frac{1}{2}} \leq AR.$$

When $|z-z_0| \geq 2R$, $|z-z_0| \sim |z-w|$ for $w \in \operatorname{supp}\varphi(\frac{\cdot-z_0}{R})$, so

$$II \leq C\left(\int_{|z-z_0|\geq 2R} \frac{1}{|z-z_0|^4}\left(\int_{\mathbb{C}}\left|\varphi(\tfrac{w-z_0}{R})\right|\,dw\right)^2 dz\right)^{\frac{1}{2}} \leq CR^2\left(\int_{r>R}\frac{1}{r^3}\,dr\right)^{\frac{1}{2}} \leq AR.$$

□

The final ingredient we need to prove Theorem 1.1 is a result by Ricci and Stein [RS87].

Theorem 2.7 (Ricci-Stein). *In $\mathbb{R}^n \times \mathbb{R}^n$, let $K(\cdot\,,\cdot)$ satisfy the following:*

(a) $K(\cdot\,,\cdot)$ is a C^1 function away from the diagonal $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y\}$,

(b) $|\nabla K(x,y)| \leq A|x-y|^{-n-1}$ for some $A \geq 0$,

(c) The operator $f \mapsto \int_{\mathbb{R}^n} K(x,y)f(y)\,dy$ initially defined on $C_0^\infty(\mathbb{R}^n)$ extends to a bounded operator on $L^2(\mathbb{R}^n)$.

Then the operator T defined by

$$T[f](x) = \int_{\mathbb{R}^n} e^{iP(x,y)}K(x,y)f(y)\,dy$$

can be extended to a bounded operator from $L^q(\mathbb{R}^n)$ to itself, with $1 < q < \infty$. The bound of this operator may depend on K, q, n and the degree d of P but is otherwise independent of the coefficients of P.

Proof (Theorem 1.1). Lemmas 2.5 and 2.6 and the Size Estimate (2.6) allow us to use the T(1) Theorem (p. 294 in [Ste93]) and to prove L^2 boundedness for S_τ with constants independent of τ. S_τ satisfies the hypotheses of Theorem 2.7, so T_τ is a bounded operator from L^q to itself with the bounds depending only on S_τ, q, and the degree of p. The bounds do not depend on the coefficients of p or on τ. $\square$

2.3 Equivalence with NIS operators

2.3.1 NIS operators on polynomial model domains in $\mathbb{C}^2$

There are different notions NIS operators (e.g. [NRSW89, NS01]). We use the definition from [NRSW89].

Definition 2.8 (Nonistropic Smoothing Operator of order m)**.** *Let*

$$T[f](z,t) = \int_{\mathbb{C}\times\mathbb{R}} T\big((z,t),(w,s)\big) f(w,s)\,dwds,$$

where $T\big((z,t),(w,s)\big)$ is a distribution which is C^∞ away from the diagonal. We shall say that T is a nonistropic smoothing operator which is smoothing of order m if there exists a family

$$T_\epsilon[f](z,t) = \int_{\mathbb{C}\times\mathbb{R}} T_\epsilon\big((z,t),(w,s)\big) f(w,s)\,dwds,$$

so that:

(a) $T_\epsilon[f] \to T[f]$ *in* $C^\infty(\mathbb{C}\times\mathbb{R})$ *as* $\epsilon \to 0$ *whenever* $f \in \mathbb{C}_c^\infty(\mathbb{C}\times\mathbb{R})$*;*

(b) Each $T_\epsilon\big((z,t),(w,s)\big) \in C^\infty\big((\mathbb{C}\times\mathbb{R})\times(\mathbb{C}\times\mathbb{R})\big)$*;*

The following two conditions hold uniformly in ϵ,

(c)

$$\left|\mathcal{X}^I T_\epsilon\big((z,t),(w,s)\big)\right| \le c_{|I|}\frac{d_{NI}(z,w,t-s)^{m-|I|}}{V\big((z,t),(w,s)\big)}, \tag{2.13}$$

where we have used the abbreviation $\mathcal{X}^I = \mathcal{X}_{i_1}\mathcal{X}_{i_2}\cdots\mathcal{X}_{I_k}$ *and* $\mathcal{X}_{i_j} = L_z, L_w, \bar{L}_z$, *or* $\bar{L}_w$*;*

(d) For each $\ell \ge 0$*, there exists an* $N = N_\ell$ *so that whenever* φ *is a smooth (bump) function supported in* $B_{NI}\big((z,t),\delta\big)$*,*

$$\left|\mathcal{X}^I T[\varphi](z,t)\right| \le C_\ell \delta^{m-\ell} \sup_{w,s} \sum_{|J|\le N_\ell} \delta^{|J|}\left|\mathcal{X}^J[\varphi](w,s)\right|, \tag{2.14}$$

whenever $|I| = \ell$*;*

(e) The same estimates hold for the operator T^**, i.e. the operator with the kernel* $\overline{T\big((w,s),(z,t)\big)}$*.*

We now generate an OPF operator T_τ from an NIS operator $\tilde{T}$ on a polynomial model domain M_p. Let $\tilde{k}(p,q)$ be the kernel of an NIS operator $\tilde{T}$. On $\mathbb{C}\times\mathbb{R}$, each kernel $\tilde{k}$ can be associated with a kernel k by setting

$$k(z,w,t-s) = \tilde{k}((z,t),(w,s)).$$

The convolution structure in t follows from the property that a polynomial domain is translation invariant in $t = \operatorname{Re} z_2$. Thus we have (for appropriate φ),

$$\tilde{T}[\varphi](z,t) = \int_{\mathbb{C}\times\mathbb{R}} \tilde{k}((z,t),(w,s))\varphi(w,s)\,dwds = \int_{\mathbb{C}\times\mathbb{R}} k(z,w,t-s)\varphi(w,s)\,dwds.$$

We set

$$K_\tau(z,w) = \int_{\mathbb{R}} e^{-i\tau t} k(z,w,t)\,dt$$

and observe we also have

$$k(z,w,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\tau} K_\tau(z,w)\,dt.$$

The integrals representing $K_\tau(z,w)$ and $k(z,w,t)$ do not necessarily converge. For a tempered distribution T and a Schwartz function φ, we know that if $\mathcal{F}$ represents the partial Fourier transform in t, by definition, $\langle \mathcal{F}T, \varphi\rangle = \langle T, \mathcal{F}\varphi\rangle$. As an integral, this corresponds to:

$$\begin{aligned}\langle \mathcal{F}T, \varphi\rangle &= \int_{\mathbb{C}\times\mathbb{R}} k(z,w,t)\int_{\mathbb{R}} e^{-it\tau}\varphi(w,\tau)\,d\tau dw dt\\ &= \int_{\mathbb{C}\times\mathbb{R}}\int_{\mathbb{R}} k(z,w,t)e^{-it\tau}\,dt\,\varphi(w,\tau)\,dwdt.\end{aligned}$$

We make sense of the second line by the string of equalities and call the integral $\int_{\mathbb{R}} k(z,w,t)e^{-it\tau}\,dt$ as being defined in the sense of Schwartz distributions. We similarly justify writing $k(z,w,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\tau}K_\tau(z,w)\,d\tau$. If one of (or both of) the kernels is actually in $L^1(\mathbb{R})$ (in t or τ), then the integral defined in the sense of Schwartz distributions agrees with the standard definition.

2.3.2 NIS Operator on $\mathbb{C}\times\mathbb{R}$ generates a family T_τ on $\mathbb{C}$

Theorem 2.9. *An NIS operator $\tilde{T}$ of order $m \le 2$ on a polynomial model domain $M_p = \{(z_1,z_2)\in\mathbb{C}^2 : \operatorname{Im} z_2 = p(z_1)\}$ generates a family of operators T_τ of order $m \le 2$ with respect to the polynomial p.*

Theorem 2.9 is proved in a series of lemmas. Two comments first, however. One, the approximation conditions imply one another since a partial Fourier transform is a continuous operator on the space of Schwartz distributions. Also, the adjoint conditions allow us to focus only k and K_τ as the computations will automatically apply to k^* and K_τ^*.

We first show that if $\tilde{k}$ is an NIS operator of order $m \leq 2$, then K_τ is the kernel for a family T_τ of operators on $\mathbb{C}$.

Lemma 2.10. *The operator T_τ has the w-cancellation condition* (2.8)*.*

Proof. Let $Y_{\tau p}^J$ be a product of $|J|$ operators of the form $Y_{\tau p}^j = Z_{\tau p},\ \bar{Z}_{\tau p},\ M_{\tau p}$ where $|J| = \ell + n$ and $n = \#\{j : Y_{\tau p}^j = M_{\tau p}\}$ and let $\varphi \in C^\infty(D(z_0, \delta))$. We have

$$K_{\tau,\epsilon}(z, w) = \int_{\mathbb{R}} e^{-i\tau t} k_\epsilon(z, w, t)\, dt,$$

so that integration by parts yields

$$\begin{aligned} Z_{\tau p} K_{\tau,\epsilon}(z, w) &= Z_{\tau p} \int_{\mathbb{R}} e^{-i\tau t} k_\epsilon(z, w, t)\, dt \\ &= \frac{\partial}{\partial z} \int_{\mathbb{R}} e^{-i\tau t} k_\epsilon(z, w, t)\, dt - \int_{\mathbb{R}} \tau \frac{\partial p}{\partial z}(z) e^{-i\tau t} k_\epsilon(z, w, t)\, dt \\ &= \int_{\mathbb{R}} e^{-i\tau t} L k_\epsilon(z, w, t)\, dt. \end{aligned}$$

Similarly, $\bar{Z}_{\tau p,z} K_{\tau,\epsilon}(z, w) = \int_{\mathbb{R}} e^{-i\tau t} \bar{L}_z k_\epsilon(z, w, t)\, dt$. Also, recalling that $\mathcal{M} f(z, w) = -i\big(t + T(w, z)\big) f(z, w)$, we have

$$M_{\tau p} K_{\tau,\epsilon}(z, w) = \int_{\mathbb{R}} e^{-i\tau(t+T(w,z))} \mathcal{M} k_\epsilon(z, w, t)\, dt.$$

Thus,

$$\int_{\mathbb{C}} Y_{\tau p}^J K_{\tau,\epsilon}(z, w) \varphi(w)\, dw = \int_{\mathbb{C}} \int_{\mathbb{R}} e^{-i\tau t} \mathcal{Y}^J k(z, w, t) \varphi(w)\, dt dw,$$

with the correspondence that if $Y_{\tau p}^j = Z_{\tau p}, \bar{Z}_{\tau p}, M_{\tau p}$, then $\mathcal{Y}^j = L, \bar{L}, \mathcal{M}$ respectively.

So,

$$
\begin{aligned}
&\int_{\mathbb{C}} Y^J_{\tau p} K_{\tau,\epsilon}(z,w)\varphi(w)\,dw = \iint_{\mathbb{C}\times\mathbb{R}} (\mathcal{Y}^J k_\epsilon)(z,w,t)e^{-i\tau t}\varphi(w)\,dtdw \\
&= \frac{c_{n+k}}{\tau^{n+k}} \iint_{\mathbb{C}\times\mathbb{R}} \left(\frac{\partial^{n+k}}{\partial t^{n+k}}\mathcal{Y}^J\right) k_\epsilon(z,w,t)e^{-i\tau t}\varphi(w)\,dtdw \\
&= \frac{c_{n+k}}{\tau^{n+k}} \iint_{\mathbb{C}\times\mathbb{R}} \left(\frac{\partial^{n+k}}{\partial t^{n+k}}\mathcal{Y}^J\right) k_\epsilon(z,w,t)e^{-i\tau t}\varphi(w)\eta(w,t)\,dtdw \\
&\quad + \frac{c_{n+k}}{\tau^{n+k}} \iint_{\mathbb{C}\times\mathbb{R}} \left(\frac{\partial^{n+k}}{\partial t^{n+k}}\mathcal{Y}^J\right) k_\epsilon(z,w,t)e^{-i\tau t}\varphi(w)(1-\eta(w,t))\,dtdw
\end{aligned}
\tag{2.15}
$$

where $\eta \in C^\infty_c(\mathbb{C}\times\mathbb{R})$ is a bump function on $B_{NI}((z,0),\delta)$. The strategy is to expand $\left(\frac{\partial^{n+k}}{\partial t^{n+k}}\mathcal{Y}^J\right) k_\epsilon(z,w,t)$ and estimate an arbitrary term for each of integral in (2.15). It is important to remember that in $\mathcal{Y}^J$, n of the terms are $\mathcal{M}$ and a L or $\bar{L}$ can hit either an $\mathcal{M}$ term or $k_\epsilon(z,w,t)$.

Expanding $\left(\frac{\partial^{n+k}}{\partial t^{n+k}}\mathcal{Y}^J\right) k_\epsilon(z,w,t)$, we see

$$
\begin{aligned}
&\frac{\partial^{n+k}}{\partial t^{n+k}}\mathcal{Y}^J k_\epsilon(z,w,t) \\
&= \frac{\partial^{n+k}}{\partial t^{n+k}}\left[\sum_{|J_0|+\cdots+|J_n|=\ell}\left(c_{|J_0|,\ldots,|J_n|}\mathcal{X}^{J_0}k_\epsilon(z,w,t)\prod_{j=1}^n(-i)\mathcal{X}^{J_j}\big(t+T(w,z)\big)\right)\right] \\
&= \sum_{|J_0|+\cdots+|J_n|=\ell} c_{|J_0|,\ldots,|J_n|}\sum_{\ell_0+\cdots+\ell_n=n+k} c_{\ell_0,\ldots,\ell_n}\frac{\partial^{\ell_0}}{\partial t^{\ell_0}}\mathcal{X}^{J_0}k_\epsilon(z,w,t)\prod_{j=1}^n\frac{\partial^{\ell_j}}{\partial t^{\ell_j}}\mathcal{X}^{J_j}\big(t+T(w,z)\big),
\end{aligned}
\tag{2.16}
$$

where $\mathcal{X}^{J_j}$ is an operator composed only of $\mathcal{X}^j = L$ and $\bar{L}$. We pick an arbitrary term from the sum and show that it has the desired bound. Taking an arbitrary term from (2.16), we estimate the integrals from (2.15) which reduces to the following two integrals:

$$
I = \left|\frac{1}{\tau^{n+k}}\iint_{\mathbb{C}\times\mathbb{R}}\frac{\partial^{\ell_0}}{\partial t^{\ell_0}}\mathcal{X}^{J_0}k_\epsilon(z,w,t)\prod_{j=1}^n\frac{\partial^{\ell_j}}{\partial t^{\ell_j}}\mathcal{X}^{J_j}\big(t+T(w,z)\big)e^{-i\tau t}\varphi(w)\eta(w,t)\,dtdw\right|
$$

and

$$
II = \left|\frac{1}{\tau^{n+k}}\iint_{\mathbb{C}\times\mathbb{R}}\frac{\partial^{\ell_0}}{\partial t^{\ell_0}}\mathcal{X}^{J_0}k_\epsilon(z,w,t)\prod_{j=1}^n\frac{\partial^{\ell_j}}{\partial t^{\ell_j}}\mathcal{X}^{J_j}\big(t+T(w,z)\big)e^{-i\tau t}\varphi(w)(1-\eta(w,t))\,dtdw\right|
$$

where $|J_0|+\cdots+|J_\ell|=\ell$ and $\ell_0+\cdots+\ell_n=n+k$. Recall from Proposition 2.4, we have

$$\left|\frac{\partial^{\ell_j}}{\partial t^{\ell_j}}\mathcal{X}^{J_j}\big(t+T(w,z)\big)\right| \le c_{\ell_j,|J_j|}\Lambda(z,d_{NI}(z,w,t))^{1-\ell_j}d_{NI}(z,w,t)^{-|J_j|}.$$

Using this fact and the cancellation condition (2.8), I has the estimate

$$\begin{aligned}
I &\le \frac{c_{|J_0|,\ell_0}}{|\tau|^{n+k}}\frac{\delta^{m-|J_0|}}{\Lambda(z,\delta)^{\ell_0}}\sup_{(w,t)}\sum_{|I|\le N_{|J_0|,\ell_0}}\delta^{|I|}\left|\mathcal{X}^I\left(e^{-i\tau t}\varphi(w)\prod_{j=1}^n\left(\frac{\partial^{\ell_j}}{\partial t^{\ell_j}}\mathcal{X}^{J_j}\big(t+T(w,z)\big)\right)\eta(w,t)\right)\right| \\
&\le \frac{c_{|J_0|,\ell_0}}{|\tau|^{n+k}}\frac{\delta^{m-|J_0|}}{\Lambda(z,\delta)^{\ell_0}}\sup_{(w,t)}\sum_{|I|\le N_{|J_0|,\ell_0}}\delta^{|I|}\sum_{|I_0|+\cdots+|I_{n+1}|=|I|}c_{I_0,\dots,I_{n+1}}\Bigg|\mathcal{X}^{I_0}\big(e^{-i\tau t}\varphi(w)\big) \\
&\qquad\times\prod_{j=1}^n\left(\mathcal{X}^{I_j}\frac{\partial^{\ell_j}}{\partial t^{\ell_j}}\mathcal{X}^{J_j}\big(t+T(w,z)\big)\right)\mathcal{X}^{I_{n+1}}\eta(w,t)\Bigg| \\
&\le \frac{c_{n,\ell,k}}{|\tau|^{n+k}}\frac{\delta^{m-|J_0|}}{\Lambda(z,\delta)^{\ell_0}}\sup_{(w,t)}\sum_{|I|\le N_{|J_0|,\ell_0}}\delta^{|I|}\sum_{|I_0|+\cdots+|I_{n+1}|=|I|}\big|\mathcal{X}^{I_0}(e^{-i\tau t}\varphi(w))\big|\Lambda(z,\delta)^{n-\ell_1-\cdots-\ell_n} \\
&\qquad\times\delta^{-(|I_1|+|J_1|+\cdots+|I_n|+|J_n|)}\delta^{-|I_{n+1}|} \\
&\le \frac{c_{n,\ell,k}}{|\tau|^{n+k}}\Lambda(z,\delta)^{-k}\delta^{m-\ell}\sup_{(w,t)}\sum_{|I_0|\le N_{|J_0|,\ell}}\delta^{|I_0|}\big|\mathcal{X}^{I_0}(e^{-i\tau t}\varphi(w))\big| \\
&= \frac{c_{n,\ell,k}}{|\tau|^{n+k}}\Lambda(z,\delta)^{-k}\delta^{m-\ell}\sup_{w}\sum_{|I_0|\le N_{|J_0|,\ell}}\delta^{|I_0|}\big|X_\tau^{I_0}\varphi(w)\big|.
\end{aligned}$$

To estimate II, we use size estimates and the support size of φ. Also, if we make the substitution $s=\mu\big(z,t+T(w,z)\big)^{-1}$, then $|\frac{ds}{dt}|\sim|\big((t+T(w,z))\mu\big(z,t+T(w,z)\big)\big)|^{-1}$. This means $|\frac{1}{s}\frac{ds}{dt}|\sim|t+T(w,z)|^{-1}$, and

$$\begin{aligned}
II &\le \frac{c_{n,\ell}}{|\tau|^{n+k}}\|\varphi\|_{L^\infty}\int_{|w-z_0|\le\delta}\int_{|t+T(w,z)|\ge\Lambda(z,\delta)}\frac{d_{NI}(z,w,t)^{m-2-|J_0|}}{\Lambda(z,d_{NI}(z,w,t))^{1+\ell_0}} \\
&\qquad\frac{\Lambda(z,d_{NI}(z,w,t))^n}{d_{NI}(z,w,t)^{|J_1|+\cdots|J_n|}\Lambda(z,d_{NI}(z,w,t))^{\ell_1+\cdots+\ell_n}}\times dtdw \\
&\le \frac{c_{n,\ell}}{|\tau|^{n+k}}\|\varphi\|_{L^\infty}\int_{|w-z_0|\le\delta}\int_{|t+T(w,z)|\ge\Lambda(z,\delta)}\mu(z,t+T(w,z))^{m-\ell-2}\frac{1}{|t+T(w,z)|^{n+k-n+1}}\,dtdw \\
&\le \frac{c_{n,\ell}}{|\tau|^{n+k}}\|\varphi\|_{L^\infty}\Lambda(z,\delta)^{-k}\delta^2\int_{|s|\le\frac{1}{\delta}}s^{1-m+\ell}\,ds \\
&\le \frac{c_{n,\ell}}{|\tau|^{n+k}}\|\varphi\|_{L^\infty}\Lambda(z,\delta)^{-k}\delta^{m-\ell}.
\end{aligned}$$

□

The proof that $K_{\tau,\epsilon}$ satisfies the size conditions (2.6) and (2.7) is broken into two lemmas. We handle the $m \leq 1$ case and the $m = 2$ case.

Lemma 2.11. *If $m \leq 1$, the kernel $K_{\tau,\epsilon}$ satisfies the size condition* (2.6).

Proof. It is enough to assume

$$Y^J_{\tau p} = M^n_{\tau p} = e^{i\tau T(w,z)} \frac{\partial^n}{\partial \tau^n} e^{-i\tau T(w,z)}$$

where $|J| = n$. Let $\eta \in C^\infty_c(\mathbb{R})$ so that $\eta \equiv 1$ on $[-1, 1]$, $0 \leq \eta \leq 1$, and $|\eta^{(n)}| \leq c_n$. Also, let $\eta_A(t) = \eta(t/A)$. We will estimate

$$\frac{\partial^n}{\partial \tau^n} \int_{\mathbb{R}} e^{-i\tau(t+T(w,z))} k_\epsilon(z, w, t)\eta_A(t)\, dt,$$

and (2.6) will follow by sending $A \to \infty$. The integral is compactly supported and the integrand is smooth, we can apply the derivatives inside of the integral. Integrating by parts $(n + k)$ times shows

$$\begin{aligned}
&c_n \left| \int_{\mathbb{R}} e^{-i\tau(t+T(w,z))} \big(t + T(w, z)\big)^n k_\epsilon(z, w, t)\eta_A\big(t + T(w, z)\big)\, dt \right| \\
&= \frac{c_{n+k}}{|\tau|^{n+k}} \left| \int_{\mathbb{R}} e^{-i\tau(t+T(w,z))} \frac{\partial^{n+k}}{\partial t^{n+k}} \Big(\big(t + T(w, z)\big)^n k_\epsilon(z, w, t)\eta_A(t + T(w, z))\Big)\, dt \right| \\
&= \frac{c_{n+k}}{|\tau|^{n+k}} \left| \int_{\mathbb{R}} e^{-i\tau(t+T(w,z))} \sum_{j=0}^{n+k} c_j \frac{\partial^j}{\partial t^j} \Big(\big(t + T(w, z)\big)^n k_\epsilon(z, w, t)\Big) \eta_A^{(n+k-j)}\big(t + T(w, z)\big)\, dt \right| \\
&\leq \frac{c_{n+k}}{|\tau|^{n+k}} \sum_{j=1}^{n+k} \Bigg[\int_{|t+T(w,z)| \leq \Lambda(z,|w-z|)} \Lambda(z, |w - z|)^{n-1-j} |w - z|^{m-2} \frac{1}{A^{n+k-j}}\, dt \\
&+ \int_{\Lambda(z,|w-z|) \leq |t+T(w,z)| \leq 2A} |t + T(w, z)|^{n-1-j} \mu(z, |t + T(w, z)|)^{m-2} \frac{1}{A^{n+k-j}} \left|\eta^{(n+k-j)}\big(\tfrac{t+T(w,z)}{A}\big)\right| dt \Bigg].
\end{aligned}$$

If $j = n + k$, then

$$\begin{aligned}
&\frac{1}{|\tau|^{n+k}} \int_{|t+T(w,z)| \le \Lambda(z,|w-z|)} \Lambda(z,|w-z|)^{n-1-(n+k)} |w-z|^{m-2}\, dt \\
&+ \frac{1}{|\tau|^{n+k}} \int_{\Lambda(z,|w-z|) \le |t+T(w,z)| \le 2A} |t+T(w,z)|^{n-1-j} \mu(z,|t+T(w,z)|)^{m-2} \frac{1}{A^{n+k-j}} \eta\big(\tfrac{t+T(w,z)}{A}\big)\, dt \\
&\le c_{n+k} \frac{|w-z|^{m-2}}{|\tau|^{n+k}\Lambda(z,|w-z|)^k} + \frac{|w-z|^{m-1}}{|\tau|^{n+k}} \int_{\Lambda(z,|w-z|) \le |t+T(w,z)|} |t+T(w,z)|^{-1-k} \mu(z,|t+T(w,z)|)^{-1}\, dt.
\end{aligned}$$

Using the substitution $s = \mu(z,|t+T(w,z)|)^{-1}$, so $|\frac{ds}{dt}| \sim \frac{1}{\mu(z,|t+T(w,z)|)|t+T(w,z)|}$,

$$\begin{aligned}
&\frac{|w-z|^{m-1}}{|\tau|^{n+k}} \int_{\Lambda(z,|w-z|) \le |t+T(w,z)|} |t+T(w,z)|^{-1-k} \mu(z,|t+T(w,z)|)^{-1}\, dt \\
&\sim \frac{|w-z|^{m-1}}{|\tau|^{n+k}} \int_{|s| \le \frac{1}{|w-z|}} \frac{1}{\Lambda(z,\frac{1}{s})^k}\, ds \\
&\le c_{n+k} \frac{|w-z|^{m-2}}{|\tau|^{n+k}\Lambda(z,|w-z|)^k}.
\end{aligned}$$

If $j < n+k$, then using the support condition of $\eta_A^{(j)}\big(t+T(w,z)\big)$ that $|t+T(w,z)| \sim A$, the estimate simplifies to

$$\begin{aligned}
&\frac{1}{|\tau|^{n+k}} \int_{|t+T(w,z)| \le \Lambda(z,|w-z|)} \Lambda(z,|w-z|)^{n-1-j} |w-z|^{m-2} \frac{1}{A^{n+k-j}}\, dt \\
&+ \frac{1}{|\tau|^{n+k}} \int_{\Lambda(z,|w-z|) \le |t+T(w,z)| \le 2A} |t+T(w,z)|^{n-1-j} \mu(z,|t+T(w,z)|)^{m-2} \frac{1}{A^{n+k-j}} \eta^{(n+k-j)}\big(\tfrac{t+T(w,z)}{A}\big)\, dt \\
&\le c_{n+k} \Lambda(z,|w-z|)^{n-j} |w-z|^{m-2} \frac{1}{A^{n+k-j}} + c_{n+k} A^{n-1-j} \mu(z,A)^{m-2} \frac{1}{A^{n+k-j+1}} \\
&\xrightarrow{A\to\infty} 0.
\end{aligned}$$

This complete the proof for $m \le 1$. □

Lemma 2.12. *If $m = 2$, the kernel $K_{\tau,\epsilon}$ satisfies the size conditions* (2.6) *and* (2.6).

Proof. As is Lemma 2.12, we can assume that

$$Y_{\tau p}^J = M_{\tau p}^n = e^{i\tau T(w,z)} \frac{\partial^n}{\partial \tau^n} e^{-i\tau T(w,z)}$$

where $|J| = n$.

We first show the case $\mu(z, \frac{1}{\tau}) \geq |w-z|$ and assume $n = 0$. Recall that $|k_\epsilon(z,w,t)| \leq \frac{c_1}{\Lambda(z,|w-z|)+|t+T(w,z)|}$ and $|\frac{\partial k_\epsilon}{\partial t}(z,w,t)| \leq \frac{c_2}{\Lambda(z,|w-z|)^2+|t+T(w,z)|^2}$. Since k_ϵ is not integrable on $\mathbb{R}$, we need to integrate by parts to obtain an estimate on $K_{\tau,\epsilon}$. However, since $|w-z|$ is small, we need to be careful to integrate by parts as few times as possible and then only for large t. Let A be a large number.

$$
\begin{aligned}
&\left|\int_{|t+T(w,z)|\leq\frac{A}{|\tau|}} e^{-i\tau t}k_\epsilon(z,w,t)\,dt\right| \leq \left|\int_{|t+T(w,z)|\leq\Lambda(z,|w-z|)} e^{-i\tau t}k_\epsilon(z,w,t)\,dt\right| \\
&\quad + \left|\int_{\Lambda(z,|w-z|)\leq|t+T(w,z)|\leq\frac{1}{|\tau|}} e^{-i\tau t}k_\epsilon(z,w,t)\,dt\right| + \left|\int_{\frac{1}{|\tau|}\leq|t+T(w,z)|\leq\frac{A}{|\tau|}} e^{-i\tau t}k_\epsilon(z,w,t)\,dt\right| \\
&\quad \lesssim 1 + \int_{\Lambda(z,|w-z|)\leq|t+T(w,z)|\leq\frac{1}{|\tau|}} \frac{1}{|t+T(w,z)|}\,dt \\
&\quad + \frac{1}{|\tau|}\left|\int_{\frac{1}{|\tau|}\leq|t+T(w,z)|\leq\frac{A}{|\tau|}} e^{-i\tau t}\frac{\partial k_\epsilon}{\partial t}(z,w,t)\,dt\right| + \frac{1}{|\tau||t|}\Bigg|_{|t+T(z,w)|=\frac{1}{\tau}}^{|t+T(z,w)|=\frac{A}{\tau}} \\
&\quad \lesssim 1 + \log\left(\frac{1/|\tau|}{\Lambda(z,|w-z|)}\right) + \frac{1}{|\tau|}\int_{\frac{1}{|\tau|}\leq|t+T(w,z)|\leq\frac{A}{|\tau|}} \frac{1}{\big(t+T(w,z)\big)^2}\,dt \\
&\quad \lesssim 1 + \log\left(\frac{1/|\tau|}{\Lambda(z,|w-z|)}\right).
\end{aligned}
$$

This is actually the estimate we are looking for since

$$
\begin{aligned}
\log\left(\frac{1/|\tau|}{\Lambda(z,|w-z|)}\right) &\sim \log\left(\inf_{j,k\geq1}\left(\frac{1}{\big(|a^z_{jk}||\tau|\big)^{1/j+k}}\frac{1}{|w-z|}\right)^{j+k}\right) \\
&\sim \inf_{j,k\geq1}\log\left(\frac{1}{\big(|a^z_{jk}||\tau|\big)^{1/j+k}}\frac{1}{|w-z|}\right)^{j+k} \\
&\sim \inf_{j,k\geq1}\log\left(\frac{1}{\big(|a^z_{jk}||\tau|\big)^{1/j+k}}\frac{1}{|w-z|}\right) \\
&\sim \log\left(\inf_{j,k\geq1}\left(\frac{1}{\big(|a^z_{jk}||\tau|\big)^{1/j+k}}\right)\frac{1}{|w-z|}\right) \sim \log\left(\frac{\mu(z,\frac{1}{\tau})}{|w-z|}\right).
\end{aligned}
$$

Also, the estimate is independent of A, so we can let $A \to \infty$.

Now assume $k \geq 1$. Let $\eta \in C_c^\infty(\mathbb{R})$, $0 \leq \eta \leq 1$, $\operatorname{supp} \eta(\cdot + T(w,z)) \subset [-2,2]$, $\eta\big(t + T(w,z)\big) = 1$ if $|t| \leq 1$, and $\eta^{(k)}\big(t + T(w,z)\big) \leq c_k$. We show the case $|w-z| \geq \mu(z, \frac{1}{\tau})$. Let $A \in \mathbb{R}$ be large. Integration by parts $n+k$ times shows:

$$\begin{aligned}
&\left| \frac{\partial^n}{\partial \tau^n} \int_{\mathbb{R}} e^{-i\tau\left(t+T(w,z)\right)} k_\epsilon(z,w,t) \eta\left(\tfrac{t+T(w,z)}{A}\right) dt \right| \\
&= \left| \sum_{j=0}^{n+k} \frac{c_j}{\tau^{n+k}} \int_{\mathbb{R}} e^{-i\tau\left(t+T(w,z)\right)} \frac{\partial^j}{\partial t^j} \left(\big(t+T(w,z)\big)^n k_\epsilon(z,w,t) \right) \frac{1}{A^{n+k-j}} \frac{d^{n+k-j}\eta}{dt^{n+k-j}} \left(\tfrac{t+T(w,z)}{A}\right) dt \right| \\
&\leq \frac{C}{|\tau|^{n+k}} \left(\sum_{j=0}^{n+k-1} A A^{n-1-j} A^{-n-k+j} + \int_{\mathbb{R}} \left| \frac{\partial^{n+k}}{\partial t^{n+k}} \Big(\big(t+T(w,z)\big)^n k_\epsilon(z,w,t) \Big) \right| dt \right) \\
&\leq \frac{C}{|\tau|^{n+k}} \Bigg(\frac{1}{A^k} + \int_{|t+T(w,z)| \leq \Lambda(z,|w-z|)} \frac{1}{\Lambda(z,|w-z|)^{k+1}} \, dt \\
&\qquad\qquad + \int_{|t+T(w,z)| \geq \Lambda(z,|w-z|)} \frac{1}{|t+T(w,z)|^{k+1}} \, dt \Bigg) \\
&\leq \frac{C}{|\tau|^{n+k}} \left(\frac{1}{A^k} + \frac{1}{\Lambda(z,|w-z|)^k} \right).
\end{aligned}$$

Sending $A \to \infty$ yields the desired estimate.

We have one estimate left to compute: the case $|w-z| < \mu(z, \frac{1}{\tau})$ and $n \geq 1$. Let A be a large number. Let $0 \leq \psi_1, \psi_2^A \leq 1$ so that $1 = \psi_1 + \psi_2^A$ on $[-A, A]$. Let $\operatorname{supp} \psi_1 \subset [-2,2]$ and $\operatorname{supp} \psi_2^A \subset \{t : |t| \in [\frac{3}{2}, 2A]\}$, and assume $|\frac{\partial^n}{\partial t^n} \psi_2^A| \leq \frac{c_n}{A^n}$ if $|t| \geq \frac{A}{2}$ and $|\frac{\partial^n \psi_1}{\partial t^n}|, |\frac{\partial^n \psi_2^A}{\partial t^n}| \leq c_n$ if $|t| \leq 2$. Since $|z-w| \leq \mu(z, \frac{1}{\tau})$, $\Lambda(z, |z-w|) \lesssim \frac{1}{\tau}$.

$$\begin{aligned}
&\left| \frac{\partial^n}{\partial \tau^n} \int_{\mathbb{R}} e^{-i\tau(t+T(w,z))} k_\epsilon(z,w,t) \Big(\psi_1\big(\tau(t+T(w,z))\big) + \psi_2\big(\tau(t+T(w,z))\big) \Big) dt \right| \\
&\leq c_n \int_{|t+T(w,z)| \leq \frac{2}{|\tau|}} |t+T(w,z)|^n |k_\epsilon(z,w,t)| \, dt \\
&+ \sum_{j=0}^{n} c_j \left| \int_{\mathbb{R}} \big(t+T(w,z)\big)^n \frac{\partial^j \psi_2^A\big(\tau(t+T(w,z))\big)}{\partial \tau^j} k_\epsilon(z,w,t) e^{-i\tau(t+T(w,z))} \, dt \right|
\end{aligned}$$

Picking an arbitrary term and integrating by parts $(n+2)$ times, we have

$$\left|\int_{\mathbb{R}} (t+T(w,z))^n \frac{\partial^j \psi_2^A(\tau(t+T(w,z)))}{\partial \tau^j} k_\epsilon(z,w,t) e^{-i\tau(t+T(w,z))}\, dt\right|$$
$$\le c_{n+2} \sum_{k=0}^{n+2} \left|\int_{\mathbb{R}} \frac{1}{|t+T(w,z)|^{n+2}} \frac{\partial^k}{\partial t^k}\Big((t+T(w,z))^n k_\epsilon(z,w,t)\Big) \tau^{n+2-k} \frac{\partial^{n+2+j-k}\psi_2^A(\tau(t+T(w,z)))}{\partial\tau^j \partial t^{n+2-k}}\, dt\right|$$

If $n+2+j-k \ge 1$, the term in the sum has support near $\frac{1}{|\tau|}$ and $\frac{A}{|\tau|}$, so it is bounded by

$$\left|\int_{\mathbb{R}} \frac{1}{|\tau|^{n+2}} \frac{\partial^k}{\partial t^k}\Big((t+T(w,z))^n k_\epsilon(z,w,t)\Big) \tau^{n+2-k} \frac{\partial^{n+2+j-k}\psi_2^A(\tau(t+T(w,z)))}{\partial\tau^j \partial t^{n+2-k}}\, dt\right|$$
$$\le \frac{c_n}{|\tau|^{n+2}} \frac{1}{|\tau|^{n-1-k}} |\tau|^{n+2-k} \frac{1}{|\tau|} + \frac{c_n}{|\tau|^{n+2}} \frac{A^{n-1-k}}{|\tau|^{n-1-k}} \frac{|\tau|^{n+2-k}}{A^{n+2-k+j}} \frac{A}{|\tau|}$$
$$\xrightarrow{A\to\infty} \frac{c_n}{|\tau|^n}.$$

Finally, if $n+2+j-k=0$, then $j=0$ and $k=n+2$ and we have the estimate

$$\left|\int_{\mathbb{R}} \frac{1}{|\tau|^{n+2}} \frac{\partial^{n+2}}{\partial t^{n+2}}\Big((t+T(w,z))^n k_\epsilon(z,w,t)\Big) \psi_2^A\big(\tau(t+T(w,z))\big)\, dt\right|$$
$$\le \frac{c_n}{|\tau|^{n+2}} \int_{|t+T(w,z)|\ge \frac{1}{2|\tau|}} \frac{1}{|t+T(w,z)|^3}\, dt = \frac{c_n}{|\tau|^n}.$$

□

Lemma 2.13. *The kernel $K_{\tau,\epsilon}$ satisfies the τ-cancellation condition* (2.9)*.*

Proof. Since $\mathcal{F}^{-1}\mathcal{F} = I$ in the sense of Schwartz distributions,

$$|\mathcal{X}^J k(z,w,t)| \le C_{|J|} \frac{\mu(z,t+T(w,z))^{m-|J|}}{V(z,\mu(z,t+T(w,z))))}$$

implies $\frac{1}{2\pi}\int_{\mathbb{R}} X_{\tau p}^J\Big(e^{i\tau t}K_{\tau,\epsilon}(z,w)\Big)\, d\tau = \mathcal{X}^J k(z,w,t)$ satisfies the same estimates. □

The proof of Theorem 2.9 is complete.

2.3.3 A family of operators T_τ on $\mathbb{C}$ generate an NIS operator $\tilde{k}$ on $\mathbb{C} \times \mathbb{R}$

Theorem 2.14. *A OPF operator T_τ of order $m \le 2$ with respect to the subharmonic, nonharmonic polynomial p generates an NIS operator $\tilde{k}$ of order $m \le 2$ on the polynomial model domain $M_p = \{(z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 = p(z_1)\}$.*

Theorem 2.14 is proven in the same matter as Theorem 2.9 and the same comments about the approximation and adjoint conditions apply.

Lemma 2.15. *The operator $\tilde{k}$ satisfies the NIS cancellation conditions* (2.14)*.*

Proof. Let $\varphi \in C_c^\infty\big(B((z,t),\delta)\big)$. Let $\eta \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp}\eta \subset [-\frac{2}{\Lambda(z,\delta)}, \frac{2}{\Lambda(z,\delta)}]$ and $\eta(\tau) = 1$ when $\tau \in [-\frac{1}{\Lambda(z,\delta)}, \frac{1}{\Lambda(z,\delta)}]$. Let $\mathcal{X}^J$ be a product of $|J|$ operators of the form of $\mathcal{X}^j = \bar{L}_z$ and L_z. Then

$$\begin{aligned}
\mathcal{X}^J \iint_{\mathbb{C}\times\mathbb{R}} k_\epsilon(z,w,t-s)\varphi(w,s)\,dwds &= \iint_{\mathbb{C}\times\mathbb{R}} \mathcal{X}^J k_\epsilon(z,w,t-s)\varphi(w,s)\,dwds \\
&= \frac{1}{2\pi}\int_{\mathbb{C}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{X}^J\big(e^{i\tau(t-s)}K_{\tau,\epsilon}(z,w)\big)\varphi(w,s)\,d\tau ds d\bar{w} \\
&= \frac{1}{2\pi}\int_{\mathbb{C}}\int_{\mathbb{R}} e^{it\tau}X^J_{\tau p}K_{\tau,\epsilon}(z,w)\varphi(w,s)\,dwd\tau \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\tau}X^J_{\tau p}\int_{\mathbb{C}} K_{\tau,\epsilon}(z,w)\varphi(w,\tau)\,dwd\tau \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\tau}X^J_{\tau p}\int_{\mathbb{C}} K_{\tau,\epsilon}(z,w)\varphi(w,\tau)\,dw\,\eta(\tau)d\tau \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}} e^{it\tau}X^J_{\tau p}\int_{\mathbb{C}} K_{\tau,\epsilon}(z,w)\varphi(w,\tau)\,dw\,(1-\eta(\tau))d\tau \\
&= I + II.
\end{aligned}$$

We estimate I and II separately. By (2.8),

$$\begin{aligned}
|I| &\leq c_{|J|}\delta^{m-|J|}\int_{\mathbb{R}}\sup_{w}\sum_{|I|\leq N_{|J|}}\delta^{|I|}\left|X^{I}_{\tau p}\Big(\varphi(w,\tau)\eta(\tau)\Big)\right|d\tau \\
&= c_{|J|}\delta^{m-|J|}\int_{\mathbb{R}}\sup_{w}\sum_{|I|\leq N_{|J|}}\delta^{|I|}\left|X^{I}_{\tau p}\Big(\varphi(w,\tau)\Big)\eta(\tau)\right|d\tau \\
&\leq c_{|J|}\delta^{m-|J|}\int_{\mathbb{R}}|\eta(\tau)|\sup_{w}\sum_{|I|\leq N_{|J|}}\delta^{|I|}\|\mathcal{X}^{I}\varphi\|_{L^1(t)}\,d\tau \\
&\leq c_{|J|}\delta^{m-|J|}\frac{1}{\Lambda(z,\delta)}\sum_{|I|\leq N_{|J|}}\delta^{|I|}\|\mathcal{X}^{I}\varphi\|_{L^\infty(\mathbb{C}\times\mathbb{R})}\Lambda(z,\delta).
\end{aligned}$$

The last line follows from Hölder's inequality and the size of $\operatorname{supp}\varphi$.

$$\begin{aligned}
|II| &= \frac{1}{2\pi}\left|\int_{\mathbb{R}}e^{it\tau}\big(1-\eta(\tau)\big)\frac{1}{\tau^2}\left(X^{J}_{\tau p}\int_{\mathbb{C}}\tau^2K_{\tau,\epsilon}(z,w)\varphi(w,\tau)\,dw\right)d\tau\right| \\
&\leq c_{|J|}\int_{|\tau|>\frac{1}{\Lambda(z,\delta)}}|\tau|^{-2}\delta^{m-|J|}\sum_{|I|\leq N_{|J|}}\delta^{|I|}\|\tau^2X^{I}_{\tau p}\varphi(w,\tau)\|_{L^\infty(w)}\,d\tau. \qquad (2.17)
\end{aligned}$$

The terms in the sum can be rewritten the more useful way:

$$\begin{aligned}
\|\tau^2(X^I)^{\#}\varphi(w,\tau)\|_{L^\infty(w)} &= \sup_{w}\left|\frac{1}{2\pi}\int_{\mathbb{R}}\tau^2X^{I}_{\tau p}e^{i\tau t}\varphi(w,t)\,dt\right| \\
&= c\sup_{w}\left|\int_{\mathbb{R}}e^{i\tau t}\left(\frac{\partial^2}{\partial t^2}\mathcal{X}^{I}\varphi(w,t)\right)\,dt\right| \\
&\leq c_2\left\|\frac{\partial^2}{\partial t^2}\mathcal{X}^{I}\varphi\right\|_{L^\infty(\mathbb{C}\times\mathbb{R})}\Lambda(z,t). \qquad (2.18)
\end{aligned}$$

Using the estimate from (2.18) in (2.17),

$$\begin{aligned}
|II| &\leq c_{|J|}\delta^{m-|J|}\int_{|\tau|>\frac{1}{\Lambda(z,\delta)}}|\tau|^{-2}\sum_{|I|\leq N_{|J|}}\delta^{|I|}\left\|\left(\frac{\partial^2}{\partial t^2}\mathcal{X}^{I}\right)\varphi(w,t)\right\|_{L^\infty(\mathbb{C}\times\mathbb{R})}\Lambda(z,\delta)\,d\tau \\
&\leq c_{|J|}\delta^{m-|J|}\sum_{|I|\leq N_{|J|}}\delta^{|I|}\Lambda(z,\delta)^2\left\|\left(\frac{\partial^2}{\partial t^2}\mathcal{X}^{I}\right)\varphi(w,t)\right\|_{L^\infty(\mathbb{C}\times\mathbb{R})} \\
&\leq c_{|J|}\delta^{m-|J|}\sum_{|I|\leq N'_{|J|}}\delta^{|I|}\left\|\mathcal{X}^{I}\varphi(w,t)\right\|_{L^\infty(\mathbb{C}\times\mathbb{R})}.
\end{aligned}$$

In the final estimate, we used the fact that $\Lambda(z,\delta)\frac{\partial}{\partial t}$ can be generated by commutators of δX terms. □

Lemma 2.16. *The operator $\tilde{k}$ has the NIS size conditions* (2.13)*.*

Proof. It is enough to find the estimate on $|k_\epsilon(z,w,t)|$. We handle the $m=2$ separately. First assume $m \le 1$. If $d_{NI}(z,w,t) = |z-w|$, then

$$\int_{\mathbb{R}} e^{i\tau t} K_{\tau,\epsilon}(z,w)\,d\tau = \frac{1}{2\pi}\int_{|\tau|\le \frac{1}{\Lambda(z,|w-z|)}} e^{i\tau t} K_{\tau,\epsilon}(z,w)\,d\tau + \frac{1}{2\pi}\int_{|\tau|\ge \frac{1}{\Lambda(z,|w-z|)}} e^{i\tau t} K_{\tau,\epsilon}(z,w)\,d\tau.$$

Estimating the first integral gives us:

$$\left|\int_{|\tau|\le \frac{1}{\Lambda(z,|w-z|)}} e^{i\tau t} K_{\tau,\epsilon}(z,w)\,d\tau\right| \le c_0 \frac{|w-z|^m}{|w-z|^2\Lambda(z,|w-z|)} = c_0 \frac{d_{NI}(z,w,t)^m}{V(z,d_{NI}(z,w,t))}.$$

The tail term is no harder: by (2.6) with $\ell = n = 0$ and $k=2$,

$$\begin{aligned}\left|\int_{|\tau|\ge \frac{1}{\Lambda(z,|w-z|)}} e^{i\tau t} K_{\tau,\epsilon}(z,w)\,d\tau\right| &\le c_2 \frac{|w-z|^m}{|w-z|^2\Lambda(z,|w-z|)^2}\int_{|\tau|\ge \frac{1}{\Lambda(z,|w-z|)}} \frac{1}{\tau^2}\,d\tau \\ &\le c_2 \frac{|w-z|^m}{|w-z|^2\Lambda(z,|w-z|)}.\end{aligned}$$

The case $d_{NI}(z,w,t) = \mu(z, t+T(w,z))$ is the τ-cancellation condition (2.9).

Now assume $m=2$. The estimate to prove is

$$|k_\epsilon(z,w,t)| \le C\frac{d_{NI}(z,w,t)^2}{V(z,d_{NI}(z,w,t))} = C\frac{1}{\Lambda(z,d_{NI}(z,w,t))}.$$

Let $\eta \in C_c^\infty(\mathbb{R})$ where $\operatorname{supp}\eta \subset [-2,2]$, $\eta(\tau)=1$ if $|\tau|\le 1$, $0\le\eta\le 1$, and $\left|\frac{\partial^k\eta}{\partial\tau^k}(\tau)\right| \le C_k$. Let $\Lambda = \Lambda(z,d_{NI}(z,w,t))$. We have

$$\begin{aligned}k_\epsilon(z,w,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\tau t}K_{\tau,\epsilon}(z,w)\,d\tau \\ &= \int_{\mathbb{R}} e^{i\tau t}K_{\tau,\epsilon}(z,w)\eta(\tau\Lambda)\,d\tau + \int_{\mathbb{R}} e^{i\tau t}K_{\tau,\epsilon}(z,w)(1-\eta(\tau\Lambda))\,d\tau \\ &= I + II.\end{aligned}$$

Before we estimate I, observe $\int_\delta^\infty \frac{\log s}{s^k}\,ds = -k\frac{\log s}{s^{k+1}} + \frac{k}{k+1}\frac{1}{s^k}$. Also, with the change of variables $s = \frac{2\mu(z,\frac{1}{\tau})}{|w-z|}$, $|\frac{\partial s}{\partial \tau}| \sim \frac{\mu(z,\frac{1}{\tau})}{|w-z|}\frac{1}{|\tau|}$ and $\Lambda(s,|w-z|) \sim \frac{1}{|\tau|}$, so

$$\begin{aligned}
I &\lesssim \int_{|\tau|\le\frac{2}{\Lambda}} \log\left(\frac{2\mu(z,\frac{1}{\tau})}{|w-z|}\right) d\tau \sim \int_{|s|\ge\frac{\mu(z,\Lambda)}{|w-z|}} \frac{\log s}{s\Lambda(z,|w-z|s)}\,ds \\
&\sim \int_{\frac{\mu(z,\Lambda)}{|w-z|}}^\infty \inf_{j,k\ge 1} \frac{1}{|a^z_{jk}||w-z|^{j+k}} \frac{\log s}{s^{j+k+1}}\,ds \\
&\lesssim \inf_{j,k\ge1} \frac{1}{|a^z_{jk}||w-z|^{j+k}} \left(\frac{\log\left(\frac{\mu(z,\Lambda)}{|w-z|}\right)}{\left(\frac{\mu(z,\Lambda)}{|w-z|}\right)^{j+k+1}} + \frac{1}{\left(\frac{\mu(z,\Lambda)}{|w-z|}\right)^{j+k}} \right) \\
&\lesssim \inf_{j,k\ge1} \frac{1}{|a^z_{jk}||w-z|^{j+k}} \frac{|w-z|^{j+k}}{\mu(z,\Lambda)^{j+k}} \sim \frac{1}{\Lambda(z,\mu(z,\Lambda))} = \frac{1}{\Lambda}.
\end{aligned}$$

To estimate II, we need to separate the cases $\Lambda = \Lambda(z,|w-z|)$ and $\Lambda = |t+T(w,z)|$. We first do the case $\Lambda = \Lambda(z,|w-z|)$. By (2.6) with $k=2$ and $\ell = n = 0$,

$$II \lesssim \int_{\frac{1}{\Lambda}}^\infty \frac{1}{\tau^2\Lambda^2}\,d\tau \sim \frac{1}{\Lambda}.$$

Now assume $\Lambda = |t+T(w,z)|$. Then

$$II \lesssim \frac{1}{\big(t+T(w,z)\big)^2} \left| \int_{|\tau|\ge\frac{1}{|t+T(w,z)|}} e^{i\tau(t+T(w,z))} \frac{\partial^2}{\partial\tau^2}\Big(e^{-i\tau T(w,z)} K_{\tau,\epsilon}(z,w)(1-\eta(\tau|t+T(w,z)|))\Big)\,d\tau \right|.$$

If both τ-derivatives are applied to $K_{\tau,\epsilon}$,

$$\begin{aligned}
&\frac{1}{\big(t+T(w,z)\big)^2} \int_{|\tau|\ge\frac{1}{|t+T(w,z)|}} \left|\frac{\partial^2}{\partial\tau^2}\Big(e^{-i\tau T(w,z)}K_{\tau,\epsilon}(z,w)\Big)\right| \big(1-\eta(\tau|t+T(w,z)|)\big)\,d\tau \\
&\qquad \sim \frac{1}{\big(t+T(w,z)\big)^2} \int_{|\tau|\ge\frac{1}{|t+T(w,z)|}} \frac{1}{\tau^2}\,d\tau \sim \frac{1}{\big(t+T(w,z)\big)}.
\end{aligned}$$

Next, if one τ-derivative is applied to $K_{\tau,\epsilon}$ and one to η, then

$$\begin{aligned}
&\frac{1}{\big(t+T(w,z)\big)} \int_{|\tau|\ge\frac{1}{|t+T(w,z)|}} \left|\frac{\partial}{\partial\tau}\Big(e^{-i\tau T(w,z)}K_{\tau,\epsilon}(z,w)\Big)\right| \eta'(\tau|t+T(w,z)|)\,d\tau \\
&\qquad \sim \frac{1}{\big(t+T(w,z)\big)} \int_{|\tau|\sim\frac{1}{|t+T(w,z)|}} \frac{1}{\tau}\,d\tau \sim \frac{1}{\big(t+T(w,z)\big)}.
\end{aligned}$$

Finally, if η receives both τ-derivatives,

$$\int_{|\tau|\geq\frac{1}{|t+T(w,z)|}} |K_{\tau,\epsilon}(z,w)|\,\eta''(\tau|t+T(w,z)|))\,d\tau \sim \int_{|\tau|\sim\frac{1}{|t+T(w,z)|}} d\tau \sim \frac{1}{\big(t+T(w,z)\big)}.$$

□

Proving Theorem 2.9 and Theorem 2.14 proves Theorem 1.2.

Chapter 3

The Heat Equation and Smoothness of the Heat Kernel

Let $S_{\tau p} : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ be the Szegö projection which is the projection of $L^2(\mathbb{C})$ onto the null space of $\bar{Z}_{\tau p}$, and denote the integral kernel of $S_{\tau p}$ as $S_{\tau p}(z, w)$. We know from [Chr91a] that $S_{\tau p} \in C^\infty(\mathbb{C} \times \mathbb{C})$ as $\bar{Z}_{\tau p} S_{\tau p} = 0$ even on the diagonal.

For the remainder of the work, we will primarily be concerned with inverting the "Laplace" operators

$$\Box_{\tau p} = -\bar{Z}_{\tau p} Z_{\tau p}$$

and

$$\widetilde{\Box}_{\tau p} = -Z_{\tau p} \bar{Z}_{\tau p}$$

via the heat semigroups $e^{-s\Box_{\tau p}}$ and $e^{-s\widetilde{\Box}_{\tau p}}$, respectively. By writing out $\Box_{\tau p}$ and $\widetilde{\Box}_{\tau p}$ explicitly (which we do in Chapter 4), we see that it requires the same analysis to analyze $\Box_{\tau p}$ for $\tau < 0$ as $\widetilde{\Box}_{\tau p}$ when $\tau > 0$. Similarly, the analysis to study $\widetilde{\Box}_{\tau p}$ for $\tau > 0$ proves the identical results for $\Box_{\tau p}$ when $\tau < 0$. As such, we can assume that $\tau > 0$. The analysis of $\widetilde{\Box}_{\tau p}$ is more subtle than that of $\Box_{\tau p}$ because in $L^2(\mathbb{C})$, $\ker \widetilde{\Box}_{\tau p} = \ker \bar{Z}_{\tau p}$ is large while $\ker \Box_{\tau p} = \{0\}$.

We define the heat operators

$$\mathcal{H}_{\tau p} = \frac{\partial}{\partial s} + \Box_{\tau p}$$

and

$$\tilde{\mathcal{H}}_{\tau p} = \frac{\partial}{\partial s} + \widetilde{\Box}_{\tau p}.$$

Given a function f defined on $\mathbb{C}$, we study the initial value problem of finding smooth $u, \tilde{u} : (0,\infty) \times \mathbb{C} \to \mathbb{C}$ so that

$$\begin{cases} \mathcal{H}_{\tau p}[u](s,z) = 0 & s > 0,\ z \in \mathbb{C} \\ \lim_{s\to 0} u(s,\cdot) = f(\cdot) & \text{with convergence in an appropriate norm.} \end{cases} \tag{3.1}$$

and

$$\begin{cases} \tilde{\mathcal{H}}_{\tau p}[\tilde{u}](s,z) = 0 & s > 0,\ z \in \mathbb{C} \\ \lim_{s\to 0} \tilde{u}(s,\cdot) = \tilde{f}(\cdot) & \text{with convergence in an appropriate norm.} \end{cases} \tag{3.2}$$

We will show that the solutions u and $\tilde{u}$ defined on $(0,\infty)\times\mathbb{C}$ are given by semigroups of operators

$$u(s,z) = e^{-s\Box_{\tau p}}[f](z)$$

and

$$\tilde{u}(s,z) = e^{-s\widetilde{\Box}_{\tau p}}[f](z).$$

We also prove the existence of functions $H_{\tau p}(s,z,w)$ and $\tilde{H}_{\tau p}(s,z,w)$ which are smooth away from the diagonal $\{(s,z,w) : s = 0 \text{ and } z = w\}$ and have the property that the heat semigroups $e^{-s\Box_{\tau p}}$ and $e^{-s\widetilde{\Box}_{\tau p}}$ can be written

$$e^{-s\Box_{\tau p}}[f](z) = \int_{\mathbb{C}} f(w) H_{\tau p}(s,z,w)\, dw$$

and

$$e^{-s\widetilde{\Box}_{\tau p}}[f](z). = \int_{\mathbb{C}} f(w) \tilde{H}_{\tau p}(s,z,w)\, dw.$$

Let α be a multiindex. We let X^α be a product of $|\alpha|$ operators of the form $X = X_1$ or X_2. Similarly, U^α is a product of $|\alpha|$ operators of the form $U = U_1$ or U_2.

3.1 Heat semigroups $e^{-s\Box_{\tau p}}$ and $e^{-s\widetilde{\Box}_{\tau p}}$ on $L^2(\mathbb{C})$

We know that $\bar{Z}_{\tau p}$ and $Z_{\tau p}$ are closed, densely defined operators on $L^2(\mathbb{C})$. As in Nagel-Stein [NS01], the spectral theorem for unbounded operators (see [Rud91]) proves:

Theorem 3.1. *The operators $\Box_{\tau p}$ and $\widetilde{\Box}_{\tau p}$ are each the infintesimal generator of a strongly continuous semigroup of bounded operators on $L^2(\mathbb{C})$, $e^{-s\Box_{\tau p}}$ and $e^{-s\widetilde{\Box}_{\tau p}}$ respectively for $s > 0$. For $f \in L^2(\mathbb{C})$, the following hold:*

(a) $\lim_{s\to 0} \|e^{-s\Box_{\tau p}} f - f\|_{L^2(\mathbb{C})} = 0$ *and* $\lim_{s\to 0} \|e^{-s\widetilde{\Box}_{\tau p}} f - f\|_{L^2(\mathbb{C})} = 0$*;*

(b) For $s > 0$, these operators are contractions, that is,

$$\|e^{-s\Box_{\tau p}} f\|_{L^2(\mathbb{C})} \leq \|f\|_{L^2(\mathbb{C})} \qquad \text{and} \qquad \|e^{-s\widetilde{\Box}_{\tau p}} f\|_{L^2(\mathbb{C})} \leq \|f\|_{L^2(\mathbb{C})};$$

(c) For $f \in \mathrm{Dom}(\Box_{\tau p})$ and $\tilde{f} \in \mathrm{Dom}(\widetilde{\Box}_{\tau p})$,

$$\|e^{-s\Box_{\tau p}} f - f\|_{L^2(\mathbb{C})} \leq s\|\Box_{\tau p} f\|_{L^2(\mathbb{C})} \qquad \text{and} \qquad \|e^{-s\widetilde{\Box}_{\tau p}} f - f\|_{L^2(\mathbb{C})} \leq s\|\widetilde{\Box}_{\tau p} f\|_{L^2(\mathbb{C})};$$

(d) For $s > 0$ and all j, $\mathrm{Range}(e^{-s\Box_{\tau p}}) \subset \mathrm{Dom}(\Box^j_{\tau p})$ and $\mathrm{Range}(e^{-s\widetilde{\Box}_{\tau p}}) \subset \mathrm{Dom}(\widetilde{\Box}^j_{\tau p})$. Also, $\Box^j_{\tau p} e^{-s\Box_{\tau p}}$ and $\widetilde{\Box}^j_{\tau p} e^{-s\widetilde{\Box}_{\tau p}}$ are bounded operators on $L^2(\mathbb{C})$ with

$$\|\Box^j_{\tau p} e^{-s\Box_{\tau p}} f\|_{L^2(\mathbb{C})} \leq \left(\frac{j}{e}\right) s^{-j} \|f\|_{L^2(\mathbb{C})}$$

and

$$\|\widetilde{\Box}^j_{\tau p} e^{-s\widetilde{\Box}_{\tau p}} f\|_{L^2(\mathbb{C})} \leq \left(\frac{j}{e}\right) s^{-j} \|f\|_{L^2(\mathbb{C})};$$

(e) For any $f \in L^2(\mathbb{C})$ and $s > 0$, the Hilbert space valued functions $u(s) = e^{-s\Box_{\tau p}} f$ and $\tilde{u}(s) = e^{-s\widetilde{\Box}_{\tau p}} f$ satisfy

$$\left(\frac{\partial}{\partial s} + \Box_{\tau p}\right) u(s) = 0$$

and

$$\left(\frac{\partial}{\partial s} + \widetilde{\Box}_{\tau p}\right) \tilde{u}(s) = 0,$$

respectively;

(f) *For every $f \in L^2(\mathbb{C})$,*

$$e^{-s\widetilde{\Box}_{\tau p}} S_{\tau p}[f] = S_{\tau p} e^{-s\widetilde{\Box}_{\tau p}}[f] = S_{\tau p}[f];$$

and consequently,

(g) *For all $f \in L^2(\mathbb{C})$*

$$e^{-s\widetilde{\Box}_{\tau p}}[f] = (I - S_{\tau p}) e^{-s\widetilde{\Box}_{\tau p}}[f] + S_{\tau p}[f] = e^{-s\widetilde{\Box}_{\tau p}}(I - S_{\tau p})[f] + S_{\tau p}[f].$$

3.2 Regularity of the Heat Kernels

For each $s > 0$, define bounded operators $H^s_{\tau p} : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ and $\tilde{H}^s_{\tau p} : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ by

$$H^s_{\tau p}[f] = e^{-s\Box_{\tau p}}[f]$$

and

$$\tilde{H}^s_{\tau p}[f] = e^{-s\Box_{\tau p}}[f].$$

The analysis of $\tilde{H}^s_{\tau p}$ is inherently more complicated than that of $H^s_{\tau p}$ because $\ker \widetilde{\Box}_{\tau p} = \ker \bar{Z}_{\tau p}$ which is large while $\ker \Box_{\tau p} = \{0\}$. Also,

$$\lim_{s \to \infty} e^{-s\widetilde{\Box}_{\tau p}} = S_{\tau p},$$

so there is no chance that

$$\int_0^\infty e^{-s\widetilde{\Box}_{\tau p}}[f](z)\,ds$$

converges. To overcome this problem, we define a related operator $\tilde{G}^s_{\tau p} : L^2(\mathbb{C}) \to L^2(\mathbb{C})$ by

$$\tilde{G}^s_{\tau p}[f] = (I - S_{\tau p})e^{-s\widetilde{\Box}_{\tau p}}[f].$$

Theorem 3.2. *There are functions* $H_{\tau p}, \tilde{G}_{\tau p} \in C^\infty\big((0,\infty) \times \mathbb{C} \times \mathbb{C}\big)$ *so that for all* $f \in L^2(\mathbb{C})$*,*

$$H^s_{\tau p}[f](z) = \int_{\mathbb{C}} H_{\tau p}(s,z,w) f(w)\,dw \tag{3.3}$$

and

$$\tilde{G}^s_{\tau p}[f](z) = \int_{\mathbb{C}} \tilde{G}_{\tau p}(s,z,w) f(w)\,dw. \tag{3.4}$$

Moreover, for each fixed $s > 0$ *and* $z \in \mathbb{C}$*, the functions* $w \mapsto H_{\tau p}(s,z,w)$ *and* $w \mapsto \tilde{G}_{\tau p}(s,z,w)$ *are in* $L^2(\mathbb{C})$*, so the integrals defined in equations* (3.3) *and* (3.4) *converge absolutely. Also,*

(a) $H_{\tau p}(s,z,w) = \overline{H_{\tau p}(s,w,z)}$ *and* $\tilde{G}_{\tau p}(s,z,w) = \overline{\tilde{G}_{\tau p}(s,w,z)}$*.*

(b) For $(s,z,w) \in (0,\infty) \times \mathbb{C} \times \mathbb{C}$*,*

$$\left(\frac{\partial}{\partial s} + \Box_{\tau p,z}\right)[H_{\tau p}](s,z,w) = \left(\frac{\partial}{\partial s} + \Box^{\#}_{\tau p,w}\right)[H_{\tau p}](s,z,w) = 0$$

and

$$\left(\frac{\partial}{\partial s} + \widetilde{\Box}_{\tau p,z}\right)[\tilde{G}_{\tau p}](s,z,w) = \left(\frac{\partial}{\partial s} + \widetilde{\Box}^{\#}_{\tau p,w}\right)[\tilde{G}_{\tau p}](s,z,w) = 0.$$

(c) For any integers $j, k \geq 0$,

$$\Box^j_{\tau p,z}(\Box^{\#}_{\tau p,w})^k H_{\tau p}(s,z,w) = \Box^{j+k}_{\tau p,z} H_{\tau p}(s,z,w) = (\Box^{\#}_{\tau p,w})^{j+k} H_{\tau p}(s,z,w)$$

and

$$\widetilde{\Box}^j_{\tau p,z}(\widetilde{\Box}^{\#}_{\tau p,w})^k \tilde{G}_{\tau p}(s,z,w) = \widetilde{\Box}^{j+k}_{\tau p,z} \tilde{G}_{\tau p}(s,z,w) = (\widetilde{\Box}^{\#}_{\tau p,w})^{j+k} \tilde{G}_{\tau p}(s,z,w).$$

(d) For all integers j and multiindices α, β, the functions

$$w \longmapsto \frac{\partial^j}{\partial s^j} X^\alpha_z U^\beta_w H_{\tau p}(s,z,w)$$

and

$$w \longmapsto \frac{\partial^j}{\partial s^j} X^\alpha_z U^\beta_w \tilde{G}_{\tau p}(s,z,w)$$

are in $L^2(\mathbb{C})$ and there is a constant $c_{j,\alpha,\beta}$ so that for $R < R_{\tau p}(z)$,

$$\Big\|\frac{\partial^j}{\partial s^j} X^\alpha_z U^\beta_w H_{\tau p}(s,z,\cdot)\Big\|_{L^2(\mathbb{C})} \leq \frac{C_{\alpha,\beta,j}}{R} s^{-\frac{\alpha+\beta}{2}-j}(1+s^{-1})$$

and

$$\Big\|\frac{\partial^j}{\partial s^j} X^\alpha_z U^\beta_w \tilde{G}_{\tau p}(s,z,\cdot)\Big\|_{L^2(\mathbb{C})} \leq \frac{C_{\alpha,\beta,j}}{R} s^{-\frac{\alpha+\beta}{2}-j}(1+s^{-1}).$$

(e) The conclusions of (d) hold with the roles of z and w interchanged.

(f) For each fixed $s > 0$, $w \in \mathbb{C}$, and any nonnegative $j \in \mathbb{Z}$, the function $z \mapsto \widetilde{\Box}^j_{\tau p,z} \tilde{G}_{\tau p}(s,z,w)$ is orthogonal to the null space of $\bar{Z}_{\tau p}$.

We know $S_{\tau p}(z,w) \in C^\infty(\mathbb{C} \times \mathbb{C})$ and $\widetilde{\Box}_{\tau p} S_{\tau p} = S_{\tau p} \widetilde{\Box}_{\tau p} = 0$. Also the decay estimates for $S_{\tau p}(z,w)$ and $R_{\tau p}(z,w)$ from [Chr91a] combined with Theorem 3.2 give us the corollary:

Corollary 3.3. *There is a function* $\tilde{H}_{\tau p} \in C^\infty((0,\infty)\times\mathbb{C}\times\mathbb{C})$ *so that*

$$\tilde{H}^s_{\tau p}[f](z) = \int_{\mathbb{C}} \tilde{H}_{\tau p}(s,z,w) f(w)\, dw.$$

Moreover, for each fixed $s > 0$ *and* $z \in \mathbb{C}$*, the function* $w \mapsto \tilde{H}_{\tau p}(s,z,w)$ *and is in* $L^2(\mathbb{C})$*, so the integral converges absolutely. Also,*

(a) $\tilde{H}_{\tau p}(s,z,w) = \overline{\tilde{H}_{\tau p}(s,w,z)}$ *and*

(b) For $(s,z,w) \in ((0,\infty)\times\mathbb{C}\times\mathbb{C})$*,*

$$\left(\frac{\partial}{\partial s} + \widetilde{\Box}_{\tau p,z}\right)[\tilde{H}_{\tau p}](s,z,w) = \left(\frac{\partial}{\partial s} + \widetilde{\Box}^{\#}_{\tau p,w}\right)[\tilde{H}_{\tau p}](s,z,w) = 0$$

(c) For any integers $j,k \geq 0$*,*

$$\widetilde{\Box}^j_{\tau p,z}(\widetilde{\Box}^{\#}_{\tau p,w})^k \tilde{H}_{\tau p}(s,z,w) = \widetilde{\Box}^{j+k}_{\tau p,z}\tilde{H}_{\tau p}(s,z,w) = (\widetilde{\Box}^{\#}_{\tau p,w})^{j+k}\tilde{H}_{\tau p}(s,z,w).$$

(d) For all integers j *and multiindices* α, β*, the functions*

$$w \longmapsto \frac{\partial^j}{\partial s^j} X^\alpha_z U^\beta_w \tilde{H}_{\tau p}(s,z,w)$$

is in $L^2(\mathbb{C})$ *and there is a constant* $C_{j,\alpha,\beta}$ *so that for* $R < R_{\tau p}(z)$*,*

$$\left\|\frac{\partial^j}{\partial s^j} X^\alpha_z U^\beta_w \tilde{H}_{\tau p}(s,z,\cdot)\right\|_{L^2(\mathbb{C})} \leq \frac{C_{\alpha,\beta,j}}{R} s^{-\frac{\alpha+\beta}{2}-j}(1+s^{-1})$$

(e) The conclusions of (d) hold with the roles of z *and* w *interchanged.*

3.3 Properties of OPF Operators

To prove Theorem 3.2, we need to establish properties of OPF operators. We follow the line of argument for NIS operators in [NRSW89, NS01]. Since we are working with a fixed polynomial p, we omit τp from subscripts when the application is clear.

Lemma 3.4. *Let A_τ and B_τ be order 0 OPF operators, and let $X = X_1$ or X_2. There exists order 0 OPF operators A_1, A_2 and B_1 and B_2 so that*

$$XA_\tau = A_1X_1 + A_2X_2$$

$$B_\tau X = X_1B_1 + X_2B_2$$

Proof. We know from results about NIS operators that $X_1^2 + X_2^2$ is invertible with an inverse K NIS smoothing of order 2. Thus, by Theorem 2.9, we can write

$$XA_\tau = \Big(XA_\tau X_1\Big)X_1 + \Big(XA_\tau K_\tau X_2\Big)X_2 = A_1X_1 + A_2X_2.$$

Similarly,

$$B_\tau X = X_1\Big(X_1K_\tau B_\tau X\Big) + X_2\Big(X_1K_\tau B_\tau X\Big) = X_1B_1 + X_1B_2.$$

□

Corollary 3.5. *Let A_τ and B_τ be order 0 OPF operators and α a multiindex where $|\alpha| = k \geq 1$. There exists a finite set I of multiindices α_i, $|\alpha_i| = k$, and order 0 OPF operators A_i and B_i so that*

$$X^\alpha A_\tau = \sum_{\alpha_i \in I} A_i X^{\alpha_i}$$

$$B_\tau X^\alpha = \sum_{\alpha_i \in I} X^{\alpha_i} B_i$$

Proof. Induction. □

Recall $[n]$ denotes the greatest integer less than or equal to n.

Proposition 3.6. *Let α be a multiindex.*

(a) If $|\alpha|$ is even, threre exists order 0 OPF operators A_τ and $\tilde{A}_\tau$ so that

$$X^\alpha = \Box_{\tau p}^{\frac{|\alpha|}{2}} A_\tau$$

and

$$(I - S_{\tau p})X^\alpha = \widetilde{\Box}_{\tau p}^{\frac{|\alpha|}{2}} \tilde{A}_\tau.$$

(b) If $|\alpha|$ is odd, there exist order 0 OPF operators A_1, $\tilde{A}_1$, A_2 and $\tilde{A}_2$ so that

$$X^\alpha = \Box_{\tau p}^{[\frac{|\alpha|}{2}]}(X_1 A_1 + X_2 A_2)$$

and

$$(I - S)X^\alpha = \widetilde{\Box}_{\tau p}^{[\frac{|\alpha|}{2}]}(X_1 \tilde{A}_1 + X_2 \tilde{A}_2)$$

Proof. Proof by induction. We know from [Chr91a] that $\Box_{\tau p}$ is invertible with inverse $G_{\tau p}$. From [NS01], and Theorem 1.2 we know $\widetilde{\Box}_{\tau p}$ has a relative fundamental solution $\tilde{G}_{\tau p}$. $G_{\tau p}$ and $\tilde{G}_{\tau p}$ are families of order 2. Suppose $|\alpha| = 2$. Then

$$X^\alpha = \Box_{\tau p} G_{\tau p} X^\alpha = \Box_{\tau p} A_\tau.$$

Similarly,

$$(I - S_{\tau p})X^\alpha = \widetilde{\Box}_{\tau p} \tilde{G}_{\tau p} X^\alpha = \widetilde{\Box}_{\tau p} \tilde{A}_\tau.$$

Now assume $|\alpha| = 2k + 2$ and $X^\alpha = X^\beta X^\gamma$ where $|\beta| = 2$ and $\gamma = 2k$. Using the induction hypothesis and Corollary 3.5, we have

$$\begin{aligned} X^\beta X^\gamma &= \Box_{\tau p} G_{\tau p} X^\beta X^\gamma = \Box_{\tau p} B_\tau X^\gamma \\ &= \Box_{\tau p} \sum_{\substack{\alpha_i \in I \\ |\alpha_i| = 2k}} X^{\alpha_i} B_i = \Box_{\tau p} \Box_{\tau p}^k \left(\sum_{\alpha_i \in I} B_{\alpha_i} B_i \right). \end{aligned}$$

Using the fact that $I - S_{\tau p} = \widetilde{\Box}_{\tau p}(I - S_{\tau p})\tilde{G}_{\tau p}$ and $\widetilde{\Box}_{\tau p} S_{\tau p} = 0$ the argument follows analogously to show $(I - S_{\tau p})X^\alpha = \widetilde{\Box}_{\tau p}^{\frac{|\alpha|}{2}} \tilde{A}_\tau$. The case $|\alpha|$ is odd follows immediately from the even case and Lemma 3.4. $\Box$

Proposition 3.7. *Let α be a multiindex.*

(a) If $|\alpha|$ is even, threre exists order 0 OPF operators B_τ and $\tilde{B}_\tau$ so that

$$X^\alpha = B_\tau \Box_{\tau p}^{\frac{|\alpha|}{2}}$$

and

$$(I - S_{\tau p})X^\alpha = \tilde{B}_\tau \widetilde{\Box}_{\tau p}^{\frac{|\alpha|}{2}}.$$

(b) If $|\alpha|$ is odd, there exist order 0 OPF operators B_1, $\tilde{B}_1$, B_2 and $\tilde{B}_2$ so that

$$X^\alpha = \Box_{\tau p}^{[\frac{|\alpha|}{2}]}(B_1 X_1 + B_2 X_2)$$

and

$$X^\alpha(I - S) = \widetilde{\Box}_{\tau p}^{[\frac{|\alpha|}{2}]}(\tilde{B}_1 X_1 \tilde{+} \tilde{B}_2 X_2)$$

(c) Alternatively, if $|\alpha|$ is odd and $X^\alpha = X^\beta X$ where $X = X_1$ or X_2, then there exists an order 0 OPF operator B_τ so that

$$X^\alpha = B_\tau \Box_{\tau p}^{\frac{|\beta|}{2}} X$$

Proof. The proof is almost identical to the proof of Proposition 3.6. $\Box$

Proposition 3.8. *Let $X = X_1$ or X_2. There is a constant C so that if $\varphi \in C_c^\infty(\mathbb{C})$, then for all $r > 0$*

$$\|X[\varphi]\|_{L^2(\mathbb{C})} \le C(r\|\Box_{\tau p}\varphi\|_{L^2(\mathbb{C})} + r^{-1}\|\varphi\|_{L^2(\mathbb{C})})$$

and

$$\|X(I - S_{\tau p})[\varphi]\|_{L^2(\mathbb{C})} \le C(r\|\widetilde{\Box}_{\tau p}\varphi\|_{L^2(\mathbb{C})} + r^{-1}\|\varphi\|_{L^2(\mathbb{C})}).$$

Proof. First, $(I - S_{\tau p})[\varphi]$ and both $X(I - S_{\tau p})[\varphi]$ and $X^2(I - S_{\tau p})[\varphi]$ are in $L^2(\mathbb{C})$. To see this, we observe that $X^2\varphi \in C_c^\infty(\mathbb{C})$. Also, the Szegö kernel is C^∞ (see [Chr91a]) and the integral is taken over $\operatorname{supp}\varphi$, a compact set. Thus, we can differentiate inside the integral and $X^2 S_{\tau p}[\varphi] \in C^\infty(\mathbb{C})$. First, note that $X^* = -X$. Then using Proposition 3.7, we compute

$$\begin{aligned}\|X(I - S_{\tau p})[\varphi]\|^2_{L^2(\mathbb{C})} &= -\Big(X^2(I - S_{\tau p})[\varphi], (I - S_{\tau p})[\varphi]\Big) \le \Big|\big(\tilde{A}\widetilde{\Box}_{\tau p}[\varphi], (I - S_{\tau p})\varphi\big)\Big| \\ &\le \|\tilde{A}\widetilde{\Box}_{\tau p}[\varphi]\|_{L^2(\mathbb{C})}\|(I - S_{\tau p})[\varphi]\|_{L^2(\mathbb{C})} \le C\|\widetilde{\Box}_{\tau p}[\varphi]\|_{L^2(\mathbb{C})}\|(I - S_{\tau p})[\varphi]\|_{L^2(\mathbb{C})} \\ &\le C\big(r^2\|\widetilde{\Box}_{\tau p}\varphi\|^2_{L^2(\mathbb{C})} + r^{-2}\|(I - S_{\tau p})\varphi\|^2_{L^2(\mathbb{C})}\big).\end{aligned}$$

Similarly,

$$\begin{aligned}\|X[\varphi]\|^2_{L^2(\mathbb{C})} &= \big(X[\varphi], X[\varphi]\big) = -\big(X^2[\varphi], \varphi\big) \\ &\le \big|\big(A\Box_{\tau p}[\varphi], \varphi\big)\big| \le C\big(r^2\|\Box_{\tau p}\varphi\|^2_{L^2(\mathbb{C})} + r^{-2}\|\varphi\|^2_{L^2(\mathbb{C})}\big).\end{aligned}$$

□

Corollary 3.9. *Let α be a multiindex. There exists a constant $C_{|\alpha|}$ so that if $\varphi \in C_c^\infty(\mathbb{C})$, then*

$$\|X^\alpha[\varphi]\|_{L^2(\mathbb{C})} \le C_\alpha \sum_{j=0}^{[\frac{|\alpha|}{2}]+1} \|\Box^j_{\tau p}[\varphi]\|_{L^2(\mathbb{C})}$$

and

$$\|X^\alpha(I - S_{\tau p})[\varphi]\|_{L^2(\mathbb{C})} \le C_\alpha \sum_{j=0}^{[\frac{|\alpha|}{2}]+1} \|\widetilde{\Box}^j_{\tau p}[\varphi]\|_{L^2(\mathbb{C})}$$

Proof. Proof by induction. The base case is Proposition 3.8 and the inductive step is a repetition of the argument in the proof of Proposition 3.8. □

We now prove our first Sobolev type theorem.

Theorem 3.10. *Let*

$$R_{\tau p}(z) = \inf_{j,k\geq 0} \frac{1}{|\tau a^z_{jk}|^{\frac{1}{j+k}}}$$

There is a constant $C > 0$ so that if $f \in C^\infty(\mathbb{C})$, $z \in \mathbb{C}$ and $0 < R < R_{\tau p}(z)$,

$$\sup_{D(z,r)} |f| \leq \frac{C}{R} \sum_{|\alpha|\leq 2} R^{|\alpha|} \|X^\alpha f\|_{L^2(D(z,2R))}.$$

If $f \in C^\infty(\mathbb{C}) \cap L^2(\mathbb{C})$, then

$$\sup_{D(z,r)} |f| \leq \frac{C}{R} \big(\|f\|_{L^2(D(z,2R))} + R^2 \|\Box_{\tau p} f\|_{L^2(D(z,2R))} \big)$$

Also, if $f \in C^\infty(\mathbb{C}) \cap L^2(\mathbb{C})$ and $f \cap \left(\ker \widetilde{\Box}_{\tau p}\right)^\perp$, then

$$\sup_{D(z,r)} |f| \leq \frac{C}{R} \Big(\|f\|_{L^2(D(z,2R))} + R^2 \|\widetilde{\Box}_{\tau p} f\|_{L^2(D(z,2R))} \Big)$$

Proof. Let $f \in C^\infty(\mathbb{C})$ and $z \in \mathbb{C}$. Let $\chi \in C^\infty_c(\mathbb{C})$, $\chi \equiv 1$ on $D(0,1)$, $0 \leq \chi \leq 1$, and $\chi(z) \equiv 0$ if $|z| \geq 2$. Let $g(z) = f(z)\chi(z - z_0)$. Then

$$\begin{aligned}
\sup_{D(z_0,1)} |f(z)| &\leq \sup_{\mathbb{C}} |g| \leq \int_{\mathbb{C}} |\hat{g}(\xi)|\, d\xi \\
&\leq \|(1+|\xi|^4)^{\frac{1}{2}} \hat{g}(\xi)\|_{L^2(\mathbb{C})} \|(1+|\xi|^4)^{-\frac{1}{2}}\|_{L^2(\mathbb{C})} \\
&\leq C\big(\|\hat{g}\|_{L^2(\mathbb{C})} + \||\xi|^2 \hat{g}\|_{L^2(\mathbb{C})}\big) \leq C\big(\|g\|_{L^2(\mathbb{C})} + \|\triangle g\|_{L^2(\mathbb{C})}\big) \\
&\leq \sum_{|\alpha|\leq 2} \|D^\alpha f\|_{L^2(D(z_0,2))}.
\end{aligned}$$

A change of variables argument shows

$$\sup_{D(z_0,R)} |f(z)| \leq \frac{C}{R} \sum_{|\alpha|\leq 2} R^{|\alpha|} \|D^\alpha f(z)\|_{L^2(D(z_0,2R))}. \tag{3.5}$$

To pass from ordinary derivatives to products of $Z_{\tau p}$ and $\bar{Z}_{\tau p}$, first observe that if

$|w-z|<R<R_{\tau p}(z)$ then

$$\left|\tau\frac{\partial^{j+k}p}{\partial z^j\partial\bar z^k}(w)\right| = \left|\tau\sum_{\substack{0\le j'\le j\\0\le k'\le k}}\frac{1}{(j-j')!(k-k')!}\frac{\partial^{j'+k'}p}{\partial z^{j'}\partial\bar z^{k'}}(z)(w-z)^{j-j'}\overline{(w-z)}^{k-k'}\right|$$
$$\le\frac{C\tau}{R^{j+k}}\sum_{\substack{0\le j'\le j\\0\le k'\le k}}\left|\frac{\partial^{j'+k'}p}{\partial z^{j'}\partial\bar z^{k'}}(z)\right|R^{j'+k'}\le\frac{C}{R^{j+k}},$$

where C does not depend on p and R. The proof of the first part of the theorem now follows easily since $\frac{\partial f}{\partial\bar z}(w)=\bar Z_{\tau p}f(w)-\tau\frac{\partial p}{\partial\bar z}(w)f(w)$, which means

$$\left|\frac{\partial f}{\partial\bar z}(w)\right|\le|\bar Z_{\tau p}f(w)|+\frac{C}{R}|f(w)|$$

and similarly for $\left|\frac{\partial f}{\partial z}(w)\right|$. Second derivatives are in the same fashion. For example,

$$\left|\frac{\partial^2 f}{\partial z\partial\bar z}(w)\right|$$
$$=\left|\bar Z_{\tau p}Z_{\tau p}[f](w)-\tau\frac{\partial p(w)}{\partial\bar z}\frac{\partial f(w)}{\partial z}+\tau^2\frac{\partial p(w)}{\partial\bar z}\frac{\partial p(w)}{\partial z}f(w)+\tau\frac{\partial p(w)}{\partial z}\frac{\partial f(w)}{\partial\bar z}+\tau\frac{\partial^2 p(w)}{\partial z\partial\bar z}f(w)\right|$$
$$\le|\bar Z_{\tau p}Z_{\tau p}[f](w)|+\frac{C}{R}|\nabla f(w)|+\frac{C}{R^2}|f|.$$

The other second derivatives of f are handled similarly. Thus, every term in (3.5) is well controlled by $Z_{\tau p}$ and $\bar Z_{\tau p}$ derivatives. The proof of the latter parts of the theorem follows from Proposition 3.7 and Proposition 3.8. □

Remark 3.11. *In Theorem 3.10, if $\frac{\partial^j p}{\partial z^j}(z)=\frac{\partial^j p}{\partial\bar z^j}(z)=0$ for $j=0,1,\dots,2m$, then $R_{\tau p}(z)=\mu(z,\frac{1}{\tau})$, a fact which will be useful later.*

Our second Sobolev type theorem is:

Theorem 3.12. *If $f\in C^\infty(\mathbb{R}\times\mathbb{C})$ and $B_{\mathbb{R}\times\mathbb{C}}\big((s,z),r\big)=\{(s',z')\in\mathbb{R}\times\mathbb{C}:|s'-s|\le r^2,|z'-z|\le r\}$, then for $0<R<R(z)$, there is a constant $C>0$ so that*

$$\sup_{B_{\mathbb{R}\times\mathbb{C}}((s,z),R)}|f|\le\frac{C}{R^2}\sum_{|\alpha|+2j\le 4}R^{2j+|\alpha|}\left\|\frac{\partial^j}{\partial s^j}X^\alpha f\right\|_{L^2(B_{\mathbb{R}\times\mathbb{C}}((s,z),2R))}.$$

If $f \in C^\infty(\mathbb{R}\times\mathbb{C})$ and $f(s,\cdot) \in L^2(\mathbb{C})$ for each s, then

$$\sup_{B_{\mathbb{R}\times\mathbb{C}}((s,z),R)} |f| \le \frac{C}{R^2}\sum_{k+j\le 2} R^{2(j+k)} \|\frac{\partial^k}{\partial s^k}\Box^j_{\tau p} f\|_{L^2(B_{\mathbb{R}\times\mathbb{C}}((s,z),2R)}.$$

Also, if $f \in C^\infty(\mathbb{R}\times\mathbb{C})$ and $f(s,\cdot) \in L^2(\mathbb{C}) \cap \left(\ker \widetilde{\Box}_{\tau p}\right)^\perp$ for each s, then

$$\sup_{B_{\mathbb{R}\times\mathbb{C}}((s,z),R)} |f| \le \frac{C}{R^2}\sum_{k+j\le 2} R^{2(j+k)} \|\frac{\partial^k}{\partial s^k}\widetilde{\Box}^j_{\tau p} f\|_{L^2(B_{\mathbb{R}\times\mathbb{C}}((s,z),2R)}.$$

Proof. The proof of Theorem 3.12 is similar to the proof of Theorem 3.10. □

3.4 Proof of Theorem 3.2

To prove Theorem 3.2, we need some a priori estimates.

Lemma 3.13. *There are constants $C_{\alpha,\beta}$ so that for any multiindices α and β, any $s > 0$, and $\varphi \in C^\infty_c(\mathbb{C})$,*

$$\|X^\alpha H^s_{\tau p}[X^\beta\varphi]\|_{L^2(\mathbb{C})} \le C_{\alpha,\beta} s^{-\frac{|\alpha|+|\beta|}{2}} \|\varphi\|_{L^2(\mathbb{C})}$$

and

$$\|X^\alpha \tilde{G}^s_{\tau p}[X^\beta\varphi]\|_{L^2(\mathbb{C})} \le C_{\alpha,\beta} s^{-\frac{|\alpha|+|\beta|}{2}} \|\varphi\|_{L^2(\mathbb{C})}.$$

Proof. We first assume that $|\alpha|$ and $|\beta|$ are even. From Proposition 3.6, there exists an order 0 OPF operator $\tilde{A}_\tau$ so that

$$\tilde{G}^s_{\tau p}[X^\beta\varphi] = e^{-s\widetilde{\Box}_{\tau p}}(I - S_{\tau p})[X^\beta\varphi] = \tilde{G}^s_{\tau p}(I - S_{\tau p})[X^\beta\varphi] = \tilde{G}^s_{\tau p}\widetilde{\Box}^{\frac{|\beta|}{2}}_{\tau p}\tilde{A}\varphi.$$

Hence, we have by Proposition 3.7 and Theorem 3.1 (d) an order zero family $\tilde{B}_\tau$ so that

$$\begin{aligned}\|X^\alpha \tilde{G}^s_{\tau p}[X^\beta\varphi]\|_{L^2(\mathbb{C})} &= \|X^\alpha(I - S_{\tau p})\widetilde{\Box}^{\frac{|\beta|}{2}}_{\tau p}\tilde{G}^s_{\tau p}[\tilde{A}_\tau\varphi]\|_{L^2(\mathbb{C})}\\ &= \|\tilde{B}_\tau\widetilde{\Box}^{\frac{|\alpha|}{2}}_{\tau p}\widetilde{\Box}^{\frac{|\beta|}{2}}_{\tau p}\tilde{G}^s_{\tau p}[\tilde{A}_\tau\varphi]\|_{L^2(\mathbb{C})}\\ &\le C_{\alpha,\beta} s^{-\frac{|\alpha|+|\beta|}{2}}\|\varphi\|_{L^2(\mathbb{C})}.\end{aligned}$$

The $|\alpha|$ and $|\beta|$ odd cases follow easily from the even case, an application of Proposition 3.6 and Proposition 3.7 and the following two arguments. One, if X is either X_1 or X_2 then from Proposition 3.8 with $r = s^{\frac{1}{2}}$

$$\|X\tilde{G}_{\tau p}\varphi\|_{L^2(\mathbb{C})} = \|X(I - S_{\tau p})\tilde{G}_{\tau p}\varphi\|_{L^2(\mathbb{C})} \leq C\Big(s^{\frac{1}{2}}\|\widetilde{\Box}_{\tau p}\tilde{G}^s_{\tau p}\varphi\|_{L^2(\mathbb{C})} + s^{-\frac{1}{2}}\|\tilde{G}^s_{\tau p}\varphi\|_{L^2(\mathbb{C})}\Big)$$
$$\leq Cs^{-\frac{1}{2}}\|\varphi\|_{L^2(\mathbb{C})}.$$

Two, since $X^* = -X$, applying the previous inequality to $\tilde{G}^s_{\tau p}X\varphi$, we have

$$\|\tilde{G}^s_{\tau p}X\varphi\|^2_{L^2(\mathbb{C})} = \Big(\tilde{G}^s_{\tau p}X\varphi, \tilde{G}^s_{\tau p}X\varphi\Big) = -\Big(\varphi, X\tilde{G}^s_{\tau p}\tilde{G}^s_{\tau p}X\varphi\Big)$$
$$\leq \|\varphi\|_{L^2(\mathbb{C})}\|X\tilde{G}^s_{\tau p}\tilde{G}^s_{\tau p}X\varphi\|_{L^2(\mathbb{C})} \leq Cs^{-\frac{1}{2}}\|\varphi\|_{L^2(\mathbb{C})}\|\tilde{G}^s_{\tau p}X\varphi\|_{L^2(\mathbb{C})}.$$

The estimates for $H^s_{\tau p}$ follow analogously. $\square$

Since $(X_j)^* = -X_j$, $j = 1, 2$, we have the immediate corollary:

Corollary 3.14. *There are constants $C_{\alpha,\beta}$ so that for any multiindices α and β, any $s > 0$, and $\varphi \in C^\infty_c(\mathbb{C})$,*

$$\|X^\alpha H^s_{\tau p}[\big(X^\beta\big)^*\varphi]\|_{L^2(\mathbb{C})} \leq C_{\alpha,\beta}s^{-\frac{|\alpha|+|\beta|}{2}}\|\varphi\|_{L^2(\mathbb{C})}$$

and

$$\|X^\alpha \tilde{G}^s_{\tau p}[\big(X^\beta\big)^*\varphi]\|_{L^2(\mathbb{C})} \leq C_{\alpha,\beta}s^{-\frac{|\alpha|+|\beta|}{2}}\|\varphi\|_{L^2(\mathbb{C})}.$$

Lemma 3.15. *For $s > 0$ and $f \in L^2(\mathbb{C})$, $H^s_{\tau p}[f]$ and $\tilde{G}^s_{\tau p}[f]$ are $C^\infty(\mathbb{C})$. Given a multiindex γ, there is a constant $C_{|\gamma|}$ so that for $z \in \mathbb{C}$ and $R < \min\{R_{\tau p}(z), 1\}$ where $R_{\tau p}(z)$ is the constant from Theorem 3.10,*

$$|X^\gamma H^s_{\tau p}[f](z)| \leq C_{|\gamma|}R^{-1}s^{-\frac{|\gamma|}{2}}(1 + s^{-1})\|f\|_{L^2(\mathbb{C})}$$

and

$$|X^\gamma \tilde{G}^s_{\tau p}[f](z)| \leq C_{|\gamma|} R^{-1} s^{-\frac{|\gamma|}{2}} (1 + s^{-1}) \|f\|_{L^2(\mathbb{C})}.$$

Proof. We can find $\varphi_n \in C_c^\infty(\mathbb{C})$ so that $\varphi_n \to f$ in $L^2(\mathbb{C})$. It follows immediately from Lemma 3.13 that $X^\gamma H^s_{\tau p}[f], X^\gamma \tilde{G}^s_{\tau p}[f] \in L^2(\mathbb{C})$, and

$$X^\gamma H^s_{\tau p}[\varphi_n] \to X^\gamma H^s_{\tau p}[f]$$

and

$$X^\gamma \tilde{G}^s_{\tau p}[\varphi_n] \to X^\gamma \tilde{G}^s_{\tau p}[f]$$

in $L^2(\mathbb{C})$, hence

$$\|X^\gamma H^s_{\tau p}[f]\|_{L^2(\mathbb{C})} \leq C_{|\gamma|} R^{-1} s^{-\frac{|\gamma|}{2}} \|f\|_{L^2(\mathbb{C})}$$

and

$$\|X^\gamma \tilde{G}^s_{\tau p}[f]\|_{L^2(\mathbb{C})} \leq C_{|\gamma|} R^{-1} s^{-\frac{|\gamma|}{2}} \|f\|_{L^2(\mathbb{C})}.$$

From these inequalities, we can show that all $Z_{\tau p}$ and $\bar{Z}_{\tau p}$ derivatives of $H^s_{\tau p}[f]$ and $\tilde{G}^s_{\tau p}[f]$ are in $L^2(\mathbb{C})$. To pass from L^2-bounds of $Z_{\tau p}$ and $\bar{Z}_{\tau p}$ derivatives to a local L^2-bound for ordinary derivatives, we can repeat the argument of Theorem 3.10. Thus, $H^s_{\tau p}[f]$ and $\tilde{G}^s_{\tau p}[f]$ are $C^\infty(\mathbb{C})$, and by Theorem 3.10,

$$\begin{aligned}\sup_{D(z,R)} |X^\gamma H^s_{\tau p}[f]| &\leq \frac{C}{R} \sum_{|\alpha| \leq 2} R^{|\alpha|} \|X^\alpha X^\gamma H^s_{\tau p}[f]\|_{L^2(\mathbb{C})} \\ &\leq \frac{C}{R} \sum_{|\alpha| \leq 2} s^{-\frac{|\alpha|+|\gamma|}{2}} \|f\|_{L^2(\mathbb{C})} \\ &\leq C_{|\gamma|} R^{-1} s^{-\frac{|\gamma|}{2}} (1 + s^{-1}) \|f\|_{L^2(\mathbb{C})}.\end{aligned}$$

Similarly,

$$\sup_{D(z,R)} |X^\gamma \tilde{G}^s_{\tau p}[f]| \leq C_{|\gamma|} R^{-1} s^{-\frac{|\gamma|}{2}} (1+s^{-1}) \|f\|_{L^2(\mathbb{C})}.$$

□

Recall the following standard fact.

Lemma 3.16. *If $x_1 \mapsto \partial^\alpha_{x_1} f(x_1,x_2)$ and $x_2 \mapsto \partial^\alpha_{x_2} f(x_1,x_2)$ are in $L^2_{\text{loc}}(\mathbb{R}^n)$ for all multiindices α, then $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.*

Proof. Smoothness is a local property, so we can assume f has compact support. Then $\partial^\alpha_{x_1} f$ and $\partial^\beta_{x_2} f$ are in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ for all multiindices α and β. By the Plancherel Theorem, if ξ and η are the transform variables of x_1 and x_2 respectively, then

$$\|\partial^\alpha_{x_1}\partial^\beta_{x_2} f\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} = c\|\xi^\alpha\eta^\beta \hat{f}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} \leq c\|(|\xi|^{\alpha+\beta} + |\eta|^{\alpha+\beta})\hat{f}\|_{L^2(\mathbb{R}^n\times\mathbb{R}^n)} < \infty.$$

The result follows by the Sobolev Embedding Theorem. □

Proof (Theorem 3.2). For $z \in \mathbb{C}$ and multiindex α, Lemma 3.15 shows that the functionals on $L^2(\mathbb{C})$ defined by

$$f(z) \mapsto \partial^\alpha H^s_{\tau p}[f](z)$$

and

$$f(z) \mapsto \partial^\alpha \tilde{G}^s_{\tau p}[f](z)$$

are bounded. By the Riesz Representation Theorem, a consequence of these facts is the existence of functions $H^{\alpha,s,z}_{\tau p}(w)$ and $\tilde{G}^{\alpha,s,z}_{\tau p}(w)$ so that

$$\partial^\alpha H^s_{\tau p}[f](z) = \int_{\mathbb{C}} H^{\alpha,s,z}_{\tau p}(w) f(w)\, dw$$

and

$$\partial^\alpha \tilde{G}^s_{\tau p}[f](z) = \int_{\mathbb{C}} \tilde{G}^{\alpha,s,z}_{\tau p}(w) f(w)\, dw.$$

Define $H^\alpha_{\tau p}(s,z,w) = H^{\alpha,s,z}_{\tau p}(w)$ and $\tilde{G}^\alpha_{\tau p}(s,z,w) = \tilde{G}^{\alpha,s,z}_{\tau p}(w)$. Also set $H_{\tau p}(s,z,w) = H^0_{\tau p}(s,z,w)$ and $\tilde{G}_{\tau p}(s,z,w) = \tilde{G}^0_{\tau p}(s,z,w)$. Then $H^\alpha_{\tau p}$ and $\tilde{G}^\alpha_{\tau p}$ are functions on $(0,\infty)\times\mathbb{C}\times\mathbb{C}$ with the property that $w \mapsto H^\alpha_{\tau p}(s,z,w)$ and $w \mapsto \tilde{G}^\alpha_{\tau p}(s,z,w)$ are in $L^2(\mathbb{C})$. Thus, we have

$$H^s_{\tau p}[f](z) = \int_{\mathbb{C}} H_{\tau p}(s,z,w) f(w)\, dw \tag{3.6}$$

and

$$\tilde{G}^s_{\tau p}[f](z) = \int_{\mathbb{C}} \tilde{G}_{\tau p}(s,z,w) f(w)\, dw, \tag{3.7}$$

and for every derivative ∂^α_z,

$$\partial^\alpha_z \left(\int_{\mathbb{C}} H_{\tau p}(s,z,w) f(w)\, dw \right) = \int_{\mathbb{C}} H^\alpha_{\tau p}(s,z,w) f(w)\, dw$$

and

$$\partial^\alpha_z \left(\int_{\mathbb{C}} \tilde{G}_{\tau p}(s,z,w) f(w)\, dw \right) = \int_{\mathbb{C}} \tilde{G}^\alpha_{\tau p}(s,z,w) f(w)\, dw.$$

We will show that $\partial^\alpha_z H_{\tau p}(s,z,w) = H^\alpha_{\tau p}(s,z,w)$ and $\partial^\alpha_z \tilde{G}_{\tau p}(s,z,w) = \tilde{G}^\alpha_{\tau p}(s,z,w)$. To do this, we use the Schwartz Kernel Theorem. Let $\varphi, \psi \in C^\infty(\mathbb{C})$. By the Schwartz Kernel Theorem,

$$\begin{aligned} \langle \partial^\alpha_z H^s_{\tau p}[\psi], \varphi \rangle_{\mathbb{C}} &= \langle (-1)^{|\alpha|} H^s_{\tau p}[\psi], \partial^\alpha_z \varphi \rangle_{\mathbb{C}} = \langle (-1)^{|\alpha|} H^s_{\tau p}, \psi \otimes \partial^\alpha_z \varphi \rangle_{\mathbb{C}\times\mathbb{C}} \\ &= \langle \partial^\alpha_z H^s_{\tau p}, \psi \otimes \varphi \rangle_{\mathbb{C}\times\mathbb{C}} = \langle (\partial^\alpha_z H^s_{\tau p})[\psi], \varphi \rangle_{\mathbb{C}}. \end{aligned}$$

Thus, we have shown

$$\partial_z^\alpha H_{\tau p}(s,z,w) = H_{\tau p}^\alpha(s,z,w) \tag{3.8}$$

and

$$\partial_z^\alpha \tilde{G}_{\tau p}(s,z,w) = \tilde{G}_{\tau p}^\alpha(s,z,w) \tag{3.9}$$

in $D'(\mathbb{C})$ and $w \mapsto \partial_z^\alpha H_{\tau p}(s,z,w)$ and $w \mapsto \partial_z^\alpha \tilde{G}_{\tau p}(s,z,w)$ are $L^2(\mathbb{C})$ functions.

Next, we know that $H_{\tau p}^s$ and $\tilde{G}_{\tau p}^s$ are self-adjoint, so

$$\int_{\mathbb{C}} H_{\tau p}^s[\psi](z)\overline{\varphi(z)}\,dz = \int_{\mathbb{C}} \psi(w)\overline{H_{\tau p}^s[\varphi](w)}\,dw$$

and

$$\int_{\mathbb{C}} \tilde{G}_{\tau p}^s[\psi](z)\overline{\varphi(z)}\,dz = \int_{\mathbb{C}} \psi(w)\overline{\tilde{G}_{\tau p}^s[\varphi](w)}\,dw.$$

As an immediate consequence of these equalities and (3.6) and (3.7), we have

$$\int_{\mathbb{C}}\int_{\mathbb{C}} H_{\tau p}(s,z,w)\psi(w)\overline{\varphi(z)}\,dwdz = \int_{\mathbb{C}}\int_{\mathbb{C}} \overline{H_{\tau p}(s,w,z)}\psi(w)\overline{\varphi(z)}\,dzdw$$

and

$$\int_{\mathbb{C}}\int_{\mathbb{C}} \tilde{G}_{\tau p}(s,z,w)\psi(w)\overline{\varphi(z)}\,dwdz = \int_{\mathbb{C}}\int_{\mathbb{C}} \overline{\tilde{G}_{\tau p}(s,w,z)}\psi(w)\overline{\varphi(z)}\,dzdw$$

It follows that $H_{\tau p}(s,z,w) = \overline{H_{\tau p}(s,w,z)}$ and $\tilde{G}_{\tau p}(s,z,w) = \overline{\tilde{G}_{\tau p}(s,w,z)}$, conclusion (a).

As a consequence of (a) and the fact that $w \mapsto H_{\tau p}^s(s,z,w)$ belongs to $L^2(\mathbb{C})$, $z \mapsto H_{\tau p}^s(s,z,w)$ belongs to $L^2(\mathbb{C})$. By equations (3.8), and (3.9), it follows that every z derivative also belongs to L^2. Thus by Lemma 3.16, $H_{\tau p}(s,z,w)$ and $\tilde{G}_{\tau p}(s,z,w)$ are $C^\infty(\mathbb{C}\times\mathbb{C})$ for fixed $s > 0$.

Next, $S_{\tau p}\widetilde{\Box}_{\tau p}^j e^{-s\widetilde{\Box}_{\tau p}}(I - S_{\tau p}) = 0$ and $S_{\tau p}$ is self-adjoint, so it follows that for all $\varphi, \psi \in C_c^\infty(\mathbb{C})$,

$$\int_{\mathbb{C}}\int_{\mathbb{C}} \widetilde{\Box}_{\tau p,z}^j \tilde{G}_{\tau p}(s,z,w)\varphi(w)\overline{S_{\tau p}[\psi](z)}\,dwdz = 0.$$

Hence for fixed s and w, $S_{\tau p}[\widetilde{\Box}_{\tau p,z}^j \tilde{G}_{\tau p}(s,\cdot,w)] = 0$. thus, $z \mapsto \widetilde{\Box}_{\tau p,z}^j \tilde{G}_{\tau p}(s,z,w)$ is in the compliment of the null space $Z_{\tau p}$. This proves (f).

We know $\Box_{\tau p}^j H_{\tau p}^s = H_{\tau p}^s \Box_{\tau p}^j$. The implication of the self-adjointness of $\Box_{\tau p}$ is that on the kernel side,

$$\Box_{\tau p,z}^j H_{\tau p}(s,z,w) = (\Box_{\tau p,w}^{\#})^j H_{\tau p}(s,z,w).$$

From this, (c) follows quickly because $\Box_{\tau p,z}^{j+k} H_{\tau p}^s = \Box_{\tau p,z}^j H_{\tau p}^s \Box_{\tau p,w}^k$. A similar argument shows $\widetilde{\Box}_{\tau p,z}^j \tilde{G}_{\tau p}(s,z,w) = (\widetilde{\Box}_{\tau p,w}^*)^j \tilde{G}_{\tau p}(s,z,w)$ which finishes the proof of (c).

Next, by Theorem 3.1 (e), $\left(\frac{\partial}{\partial s} + \Box_{\tau p}\right)\left[H_{\tau p}^s[f]\right](z) = 0$ and $\left(\frac{\partial}{\partial s} + \widetilde{\Box}_{\tau p}\right)\left[\tilde{G}_{\tau p}^s[f]\right](z) = (\frac{\partial}{\partial s} + \Box_{\tau p})(I - S_{\tau p})\left[e^{-s\widetilde{\Box}_{\tau p}}[f]\right](z) = 0$. Fixing $z \in \mathbb{C}$, integration against test functions in $(0,\infty)\times\mathbb{C}$ shows that in $\mathcal{D}'\big((0,\infty)\times\mathbb{C}\big)$,

$$\begin{aligned}
0 &= \left\langle \left(\frac{\partial}{\partial s} + \Box_{\tau p}\right)\left[H_{\tau p}^s[f]\right](z), \varphi \right\rangle \\
&= \iint_{(0,\infty)\times\mathbb{C}} H_{\tau p}(s,z,w)\left(-\frac{\partial}{\partial s} + \Box_{\tau p,z}^{\#}\right)\varphi(s,z)f(w)\,dwds \\
&= \iint_{(0,\infty)\times\mathbb{C}} \left(\frac{\partial}{\partial s} + \Box_{\tau p,z}\right) H_{\tau p}(s,z,w)\varphi(s,z)f(w)\,dwds.
\end{aligned}$$

Similarly, $\iint_{(0,\infty)\times\mathbb{C}}(\frac{\partial}{\partial s} + \widetilde{\Box}_{\tau p,z})\tilde{G}_{\tau p}(s,z,w)\varphi(s,z)f(w)\,dwds = 0$. Thus, $\frac{\partial}{\partial s}H_{\tau p}(s,z,w) = -\Box_{\tau p,z}H_{\tau p}(s,z,w)$ and $\frac{\partial}{\partial s}\tilde{G}_{\tau p}(s,z,w) = -\widetilde{\Box}_{\tau p,z}\tilde{G}_{\tau p}(s,z,w)$. Then we have

$$\frac{\partial^2}{\partial s^2}H_{\tau p}(s,z,w) = -\frac{\partial}{\partial s}\Box_{\tau p,z}H_{\tau p}(s,z,w) = -\Box_{\tau p,z}\frac{\partial}{\partial s}H_{\tau p}(s,z,w) = \Box_{\tau p,z}^2 H_{\tau p}(s,z,w).$$

Iterating this argument shows (with an identical argument for $\tilde{G}_{\tau p}(s,z,w)$)

$$\frac{\partial^j}{\partial s^j}H_{\tau p}(s,z,w) = (-1)^j \Box_{\tau p,z}^j H_{\tau p}(s,z,w) \tag{3.10}$$

and

$$\frac{\partial^j}{\partial s^j}\tilde{G}_{\tau p}(s,z,w) = (-1)^j \widetilde{\Box}^j_{\tau p,z}\tilde{G}_{\tau p}(s,z,w) \tag{3.11}$$

We know, however, that $\Box^j_{\tau p,z}H_{\tau p}(s,z,w)$ and $\widetilde{\Box}^j_{\tau p,z}\tilde{G}_{\tau p}(s,z,w)$ are both $L^2_{\text{loc}}\big((0,\infty)\times\mathbb{C}\times\mathbb{C}\big)$. As before, this is enough to show $H_{\tau p}, \tilde{G}_{\tau p} \in C^\infty\big((0,\infty)\times\mathbb{C}\times\mathbb{C}\big)$. In particular, (3.10) and (3.11) hold in the classical sense, so (b) is proved.

For α, β, and j, Lemma 3.13 shows that there is a constant $C_{\alpha,\beta,j}$ so that for $\varphi \in C^\infty_c(\mathbb{C})$,

$$\Big\| X^\alpha \Box^j_{\tau p} H^s_{\tau p}\Big[X^\beta[\varphi]\Big]\Big\|_{L^2(\mathbb{C})} \leq C_{\alpha,\beta} s^{-\frac{|\alpha|+|\beta|}{2}-j}\|\varphi\|_{L^2(\mathbb{C})}$$

and

$$\Big\| X^\alpha \widetilde{\Box}^j_{\tau p} \tilde{G}^s_{\tau p}\Big[X^\beta[\varphi]\Big]\Big\|_{L^2(\mathbb{C})} \leq C_{\alpha,\beta} s^{-\frac{|\alpha|+|\beta|}{2}-j}\|\varphi\|_{L^2(\mathbb{C})}.$$

Then by Theorem 3.10, for $R < R_{\tau p}(z)$,

$$\begin{aligned}\sup_{D(z,R)}\big|X^\alpha \Box^j_{\tau p} H^s_{\tau p}\big[X^\beta[\varphi]\big]\big| &\leq \frac{C}{R}\sum_{|\gamma|\leq 2} R^{|\gamma|}\Big\|X^{\alpha+\gamma}\widetilde{\Box}^j_{\tau p}\tilde{G}^s_{\tau p}\Big[X^\beta[\varphi]\Big]\Big\|_{L^2(\mathbb{C})}\\ &\leq \frac{C}{R}s^{-\frac{\alpha+\beta}{2}-j}(1+s^{-1})\|\varphi\|_{L^2(\mathbb{C})}.\end{aligned}$$

Similarly,

$$\sup_{D(w,R)}\big|X^\alpha \widetilde{\Box}^j_{\tau p}\tilde{G}^s_{\tau p}\big[X^\beta[\varphi]\big]\big| \leq \frac{C}{R}s^{-\frac{\alpha+\beta}{2}-j}(1+s^{-1})\|\varphi\|_{L^2(\mathbb{C})}.$$

Also, since $H_{\tau p}(s,z,w) \in C^\infty\big((0,\infty)\times\mathbb{C}\times\mathbb{C}\big)$,

$$\begin{aligned}X^\alpha \Box^j_{\tau p} H^s_{\tau p}[X^\beta\varphi](z) &= \int_{\mathbb{C}} X^\alpha_z \Box^j_{\tau p} H_{\tau p}(s,z,w) X^\beta_w \varphi(w)\,dw\\ &= \int_{\mathbb{C}} X^\alpha_z U^\beta_w \Box^j_{\tau p} H_{\tau p}(s,z,w)\varphi(w)\,dw\\ &= (-1)^j\int_{\mathbb{C}}\frac{\partial^j}{\partial s^j}X^\alpha_z U^\beta_w H_{\tau p}(s,z,w)\varphi(w)\,dw.\end{aligned}$$

From the reverse Hölder inequality and our previous estimate,

$$\left(\int_{\mathbb{C}}\left|\frac{\partial^j}{\partial s^j}X_z^\alpha U_w^\beta H_{\tau p}(s,z,w)\right|^2\,dw\right)^{\frac{1}{2}} \leq \frac{C_{\alpha,\beta,j}}{R}s^{-\frac{\alpha+\beta}{2}-j}(1+s^{-1})$$

and similarly for $\frac{\partial^j}{\partial s^j}X_z^\alpha(X_w)^\beta\tilde{G}_{\tau p}(s,z,w)$. This is (d) of the Theorem. From (a), we can interchange the roles of z and w to prove (e). This proves the theorem. □

3.5 Fundamental Solutions for $\mathcal{H}_{\tau p}$ and $\tilde{\mathcal{H}}_{\tau p}$ on $\mathbb{R}\times\mathbb{C}$

3.5.1 Distributions on $\mathbb{R}\times\mathbb{C}$

3.5.2 A Fundamental Solution for $\mathcal{H}_{\tau p}$ and a Relative Fundamental Solution for $\tilde{\mathcal{H}}_{\tau p}$ on $\mathbb{R}\times\mathbb{C}$

Define distributions $H^z_{\tau p}$, $\tilde{H}^z_{\tau p}$, and $\tilde{G}^z_{\tau p}$ on $\mathbb{R}\times\mathbb{C}$ by

Definition 3.17. *For $\psi \in C_c^\infty(\mathbb{R}\times\mathbb{C})$, set*

$$\langle H^z_{\tau p},\psi\rangle = \lim_{\epsilon\to 0}\int_\epsilon^\infty\int_{\mathbb{C}} H_{\tau p}(s,z,w)\psi(s,w)\,dwds,$$
$$\langle \tilde{H}^z_{\tau p},\psi\rangle = \lim_{\epsilon\to 0}\int_\epsilon^\infty\int_{\mathbb{C}} \tilde{H}_{\tau p}(s,z,w)\psi(s,w)\,dwds,$$

and

$$\langle \tilde{G}^z_{\tau p},\psi\rangle = \lim_{\epsilon\to 0}\int_\epsilon^\infty\int_{\mathbb{C}} \tilde{G}_{\tau p}(s,z,w)\psi(s,w)\,dwds,$$

The kernels of these distributions are

$$H^z_{\tau p}(s,w) = \begin{cases} H_{\tau p}(s,z,w) & \text{if } s > 0 \\ 0 & \text{if } s \le 0 \end{cases}$$

$$\tilde{H}^z_{\tau p}(s,w) = \begin{cases} \tilde{H}_{\tau p}(s,z,w) & \text{if } s > 0 \\ 0 & \text{if } s \le 0 \end{cases}$$

and

$$\tilde{G}^z_{\tau p}(s,w) = \begin{cases} \tilde{G}_{\tau p}(s,z,w) & \text{if } s > 0 \\ -S_{\tau p}(z,w) & \text{if } s \le 0 \end{cases}$$

For the last equality, it is helpful to recall that $\tilde{G}_{\tau p}(s,z,w) = \tilde{H}_{\tau p}(s,z,w) - S_{\tau p}(z,w)$. Also, to prove that $\tilde{G}^z_{\tau p}$ exists, it is enough to show that $\tilde{H}^z_{\tau p}$ exists.

We are going to use Theorem 3.1 and Theorem 3.10 to obtain pointwise bounds on $|e^{-s\Box_{\tau p}} f - f|$ and $|e^{-s\widetilde{\Box}_{\tau p}} g - g|$.

Lemma 3.18. *There is a constant depending only on the degree of p so that if $f \in \mathrm{Dom}(\Box^j_{\tau p})$ and $g \in \mathrm{Dom}(\widetilde{\Box}^j_{\tau p})$ for $j \le 2$ then for any $z \in \mathbb{C}$ and $0 < R < R_{\tau p}(z)$ and $z \in \mathbb{C}$,*

$$\sup_{D(z,R)} \left| f(w) - e^{-s\Box_{\tau p}}[f](w) \right| \le C\frac{s}{R}\left(R^2 \|\Box_{\tau p}[f]\|_{L^2(\mathbb{C})} + R^4 \|\Box^2_{\tau p}[f]\|_{L^2(\mathbb{C})} \right)$$

and

$$\sup_{D(z,R)} \left| g(w) - e^{-s\widetilde{\Box}_{\tau p}}[g](w) \right| \le C\frac{s}{R}\left(R^2 \|\widetilde{\Box}_{\tau p}[f]\|_{L^2(\mathbb{C})} + R^4 \|\widetilde{\Box}^2_{\tau p}[f]\|_{L^2(\mathbb{C})} \right).$$

Proof. Let $g \in \mathrm{Dom}(\widetilde{\Box}_{\tau p}^j)$, $0 \le j \le 2$. Let $\tilde{g} = (I - S_{\tau p})[g]$. Then $\tilde{g} \in \mathrm{Dom}(\widetilde{\Box}_{\tau p}^j)$, $0 \le j \le 2$. Using Theorem 3.1, we have

$$\begin{aligned}\tilde{g} - e^{-s\widetilde{\Box}_{\tau p}}\tilde{g} &= (I - S_{\tau p})g - e^{-s\widetilde{\Box}_{\tau p}}(I - S_{\tau p})[g] \\ &= g - e^{-s\widetilde{\Box}_{\tau p}}[g] - (e^{-s\widetilde{\Box}_{\tau p}}S_{\tau p}[g] - S_{\tau p}[g]) = g - e^{-s\widetilde{\Box}_{\tau p}}[g].\end{aligned}$$

Thus, by Theorem 3.10 and Theorem 3.1 (c) and the fact $\tilde{g} - e^{-s\widetilde{\Box}_{\tau p}}g$ is orthogonal to $\ker(\widetilde{\Box}_{\tau p})$, we have

$$\begin{aligned}\sup_{D(z,R)}\left|g(w) - e^{-s\widetilde{\Box}_{\tau p}}[g](w)\right| &= \sup_{D(z,R)}\left|\tilde{g}(w) - e^{-s\widetilde{\Box}_{\tau p}}[\tilde{g}](w)\right| \\ &\le C\frac{1}{R}\sum_{j=0}^{1} R^{2j}\left\|\widetilde{\Box}_{\tau p}^j[\tilde{g}] - \widetilde{\Box}_{\tau p}^j[e^{-s\widetilde{\Box}_{\tau p}}\tilde{g}]\right\|_{L^2(\mathbb{C})} \\ &= \frac{C}{R}\sum_{j=0}^{1} R^{2j}\left\|(I - e^{-s\widetilde{\Box}_{\tau p}})[\widetilde{\Box}_{\tau p}^j g]\right\|_{L^2(\mathbb{C})} \\ &\le C\frac{s}{R}\sum_{j=0}^{1} R^{2j}\|\widetilde{\Box}_{\tau p}^{j+1}[g]\|_{L^2(\mathbb{C})}.\end{aligned}$$

A similar (but simpler) arguemnt shows the analogous result with $\Box_{\tau p}$. $\square$

Lemma 3.19. *For each $z \in \mathbb{C}$, the limits $H_{\tau p}^z$, $\tilde{H}_{\tau p}^z$, and $\tilde{G}_{\tau p}^z$ exist and define distributions on $\mathbb{R} \times \mathbb{C}$.*

Proof. Let $\psi \in C_c^\infty(\mathbb{R} \times \mathbb{C})$. Then there is a closed, bounded interval $I \subset \mathbb{R}$ and a compact set $K \subset \mathbb{C}$ so that $\operatorname{supp}\psi \in I \times K$. Set $\psi_s(z) = \psi(s,z)$. Then $\{\psi_s\} \subset C_c^\infty(\mathbb{R})$ with each element having support in K. If $0 < \epsilon_1 < \epsilon_2$, then

$$\begin{aligned}\int_{\epsilon_1}^{\epsilon_2} H_{\tau p}(s,z,w)\psi(s,w)\,dw\,ds &= \int_{\epsilon_1}^{\epsilon_2} e^{-s\Box_{\tau p}}[\psi_s](z)\,ds \\ &= \int_{\epsilon_1}^{\epsilon_2} e^{-s\Box_{\tau p}}[\varphi_s](z) - \psi(s,z)\,ds + \int_{\epsilon_1}^{\epsilon_2} \psi(s,z)\,ds.\end{aligned}$$

From Lemma 3.18 and Hölder's inequality, we have (with $R < R_{\tau p}(z)$),

$$\left|\int_{\epsilon_1}^{\epsilon_2}\int_{\mathbb{C}} H_{\tau p}(s,z,w)\psi(s,w)\,dwds\right| \leq C\frac{\epsilon_2}{R}\sum_{j=0}^{1} R^{2j}\int_0^{\epsilon_2} \|\Box_{\tau p}^{j+1}[\psi_s]\|_{L^2(\mathbb{C})}\,ds + \epsilon_2\|\psi\|_{L^\infty(\mathbb{R}\times\mathbb{C})}$$
$$\leq C\frac{\epsilon_2^{\frac{3}{2}}}{R}\sum_{j=0}^{1}\|\Box_{\tau p}^{j+1}[\psi]\|_{L^2(\mathbb{R}\times\mathbb{C})} + \epsilon_2\|\psi\|_{L^\infty(\mathbb{R}\times\mathbb{C})}.$$

These last terms go to 0 as $\epsilon_2 \to 0$, so the limit defining $H_{\tau p}^z$ exist. Similarly, the limits defining $\tilde{H}_{\tau p}^z$ and $\tilde{G}_{\tau p}^z$ exist. □

Theorem 3.20. *In* $\mathcal{D}'(\mathbb{R}\times\mathbb{C})$,

$$(\partial_s + \Box_{\tau p,w}^{\#})(H_{\tau p}^z) = \delta_0\otimes\delta_z \qquad \text{and} \qquad (\partial_s + \widetilde{\Box}_{\tau p,w}^{\#})(\tilde{H}_{\tau p}^z) = \delta_0\otimes\delta_z.$$

Proof. Let $\psi \in C_c^\infty(\mathbb{R}\times\mathbb{C})$. Then

$$\langle(\partial_s + \Box_{\tau p,w}^{\#})(H_{\tau p}^z),\psi\rangle = \langle H_{\tau p}^z, (-\partial_s + \Box_{\tau p,w}^{\#})\psi\rangle$$
$$= -\lim_{\epsilon\to 0}\int_\epsilon^\infty\int_{\mathbb{C}} H_{\tau p}(s,z,w)\partial_s\psi(s,w)\,dwds + \lim_{\epsilon\to 0}\int_\epsilon^\infty\int_{\mathbb{C}} H_{\tau p}(s,z,w)\Box_{\tau p,w}\psi(w,s)\,dwds.$$

Since s is bounded away from 0 and $H_{\tau p} \in C^\infty\big((0,\infty)\times\mathbb{C}\times\mathbb{C}\big)$, the first term yields

$$-\int_\epsilon^\infty\int_{\mathbb{C}} H_{\tau p}(s,z,w)\frac{\partial\psi}{\partial s}(s,w)\,dwds$$
$$= -\int_\epsilon^\infty \frac{\partial}{\partial s}\int_{\mathbb{C}} H_{\tau p}(s,z,w)\psi(s,w)\,dwds + \int_\epsilon^\infty\int_{\mathbb{C}}\frac{\partial}{\partial s}H_{\tau p}(s,z,w)\psi(s,w)\,dwds$$
$$= \int_{\mathbb{C}} H_{\tau p}(\epsilon,z,w)\psi(\epsilon,w)\,dw + \int_\epsilon^\infty\int_{\mathbb{C}}\frac{\partial}{\partial s}H_{\tau p}(s,z,w)\psi(s,w)\,dwds.$$

Also,

$$\int_\epsilon^\infty\int_{\mathbb{C}} H_{\tau p}(s,z,w)\Box_{\tau p,w}\psi(s,w)\,dwds = \int_\epsilon^\infty\int_{\mathbb{C}}\Box_{\tau p,w}^{\#}H_{\tau p}(s,z,w)\psi(s,w)\,dwds.$$

Using Theorem 3.2 (b) and adding our equalities together, we have

$$\begin{aligned}
&\int_{\epsilon}^{\infty}\int_{\mathbb{C}} -H_{\tau p}(s,z,w)\partial_s\psi(s,w) + H_{\tau p}(s,z,w)\Box_{\tau p,w}\psi(s,w)\,dwds \\
&\qquad = \int_{\mathbb{C}} H_{\tau p}(\epsilon,z,w)\psi(\epsilon,w)\,dw + \int_{\epsilon}^{\infty}\int_{\mathbb{C}}^{\infty} (\partial_s + \Box^{\#}_{\tau p,w})H_{\tau p}(s,z,w)\psi(s,w)\,dwds \\
&\qquad = \int_{\mathbb{C}} H_{\tau p}(\epsilon,z,w)\psi(\epsilon,w)\,dw.
\end{aligned}$$

Hence

$$\begin{aligned}
\langle(\partial_s + \Box^{\#}_{\tau p,w})[H^z_{\tau p}],\psi\rangle &= \lim_{\epsilon\to 0}\int_{\mathbb{C}} H_{\tau p}(\epsilon,z,w)\psi(\epsilon,w)\,dw \\
&= \lim_{\epsilon\to 0} e^{-\epsilon\Box_{\tau p}}[\psi_\epsilon](z) = \psi(0,z) = \langle\delta_0\otimes\delta_z,\psi\rangle.
\end{aligned}$$

The identical argument shows $(\partial_s + \widetilde{\Box}^{\#}_{\tau p,w})[\tilde{H}^z_{\tau p}] = \delta_0\times\delta_z$. $\Box$

Chapter 4

Estimates on $H_{\tau p}(s,z,w)$

In this chapter, we prove Theorem 1.3, the result concerning pointwise estimates of $|X_z^I U_w^J H_{\tau p}(s,z,w)|$. We begin the chapter with a study of how the heat kernel behaves under scaling.

4.1 Scaling and the Heat Kernel

The structure of $\square_{\tau p}$ is critical in this section. Expanding $\square_{\tau p}$, we have

$$\begin{aligned}\square_{\tau p} &= -\left(\frac{\partial}{\partial \bar{z}} + \tau\frac{\partial p}{\partial \bar{z}}\right)\left(\frac{\partial}{\partial z} - \tau\frac{\partial p}{\partial z}\right) \\ &= -\frac{\partial^2}{\partial z \partial \bar{z}} + \tau\frac{\partial^2 p}{\partial z\partial \bar{z}} + \tau^2\frac{\partial p}{\partial z}\frac{\partial p}{\partial \bar{z}} + \tau\left(\frac{\partial p}{\partial z}\frac{\partial}{\partial \bar{z}} - \frac{\partial p}{\partial \bar{z}}\frac{\partial}{\partial z}\right) && (4.1) \\ &= -\frac{1}{4}\triangle + \frac{1}{4}\tau\triangle p + \frac{\tau^2}{4}|\nabla p|^2 + \frac{i}{2}\tau\left(\frac{\partial p}{\partial x_1}\frac{\partial}{\partial x_2} - \frac{\partial p}{\partial x_2}\frac{\partial}{\partial x_1}\right) && (4.2)\end{aligned}$$

Let $p_0(w) = p(w)$ and fix $z_0 \in \mathbb{C}$. Let $p_1(w) = p_0(w + z_0)$.

Proposition 4.1.

$$H_{\tau p_0}(s, z + z_0, w + z_0) = H_{\tau p_1}(s, z, w).$$

Proof. Fix $z_0 \in \mathbb{C}$/ Let $A_{z_0}[f](z) = f(z - z_0)$. A_{z_0} is an isometry on $L^2(\mathbb{C})$, and

$$\begin{aligned}
\mathcal{H}_{\tau p_1}[f](z) &= -\frac{1}{4}\triangle f(z) + \frac{\tau}{4}\triangle p_0(z+z_0)f(z) + \frac{\tau^2}{4}|\nabla p_0(z+z_0)|^2 f(z) \\
&\quad + \frac{i}{2}\tau\left(\frac{\partial p_0}{\partial x_1}(z+z_0)\frac{\partial f}{\partial x_2}(z) - \frac{\partial p_0}{\partial x_2}(z+z_0)\frac{\partial f}{\partial x_1}(z)\right) \\
&= A_{z_0}^{-1}\Bigg[-\frac{1}{4}\triangle f(z-z_0) + \frac{\tau}{4}\triangle p_0(z)f(z-z_0) + \frac{\tau^2}{4}|\nabla p_0(z)|^2 f(z-z_0) \\
&\quad + \frac{i}{2}\tau\left(\frac{\partial p_0}{\partial x_1}(z)\frac{\partial f}{\partial x_2}(z-z_0) - \frac{\partial p_0}{\partial x_2}(z)\frac{\partial f}{\partial x_1}(z-z_0)\right)\Bigg] \\
&= A_{z_0}^{-1}\mathcal{H}_{\tau p_0}A_{z_0}[f](z).
\end{aligned}$$

Also, if $\psi \in C_c^\infty(\mathbb{C}\times\mathbb{R})$,

$$A_{z_0}^{-1}(\delta_0\otimes\delta_z)A_{z_0}\psi(s,w) = A_{z_0}^{-1}\psi(0,z-z_0) = \psi(0,z),$$

and

$$\begin{aligned}
A_{z_0}^{-1}(\delta_0\otimes\delta_z)A_{z_0}\psi(s,w) &= A_{z_0}^{-1}\mathcal{H}_{\tau p_0}\int_0^\infty\int_{\mathbb{C}} H_{\tau p_0}(s,z,w)\psi(s,w-z_0)\,dwds \\
&= A_{z_0}^{-1}\int_0^\infty\int_{\mathbb{C}}\left(\frac{\partial}{\partial s} + \Box_{\tau p_0,z}\right)H_{\tau p_0}(s,z,w)\psi(s,w-z_0)\,dwds \\
&= A_{z_0}^{-1}\int_0^\infty\int_{\mathbb{C}}\left(\frac{\partial}{\partial s} + \Box^{\#}_{\tau p_0,w}\right)H_{\tau p_0}(s,z,w)\psi(s,w-z_0)\,dwds \\
&= A_{z_0}^{-1}\int_0^\infty\int_{\mathbb{C}}\left(\frac{\partial}{\partial s} + \Box^{\#}_{\tau p_1,w}\right)H_{\tau p_0}(s,z,w+z_0)\psi(s,w-z_0)\,dwds \\
&= \int_0^\infty\int_{\mathbb{C}}\left(\frac{\partial}{\partial s} + \Box^{\#}_{\tau p_1,w}\right)H_{\tau p_0}(s,z+z_0,w+z_0)\psi(s,w-z_0)\,dwds.
\end{aligned}$$

Thus, $H_{\tau p_0}(s,z+z_0,w+z_0) = H_{\tau p_1}(s,z,w)$. □

The same idea but longer calculations (which we will show) prove:

Proposition 4.2. *If* $z_0 \in \mathbb{C}$ *and*

$$p_2^{z_0}(w) = \sum_{j,k\geq 1} a_{jk}^{z}(w-z_0)^j\overline{(w-z_0)}^k,$$

then

$$H_{\tau p_2^{z_0}}(s,z,w) = e^{\tau i T(w,z_0)} H_{\tau p_1}(s,z_0,w).$$

Proof. Let $U_z[f](s,w) = e^{i\tau T(w,z)} f(s,w)$. We use the following facts: T is harmonic in each variable (so $\frac{\partial^2 T}{\partial w \partial \bar{w}} = 0$),

$$\frac{\partial p}{\partial w}(w) + i\frac{\partial T}{\partial w}(w,z) = \sum_{j,k\geq 1} j a_{jk}^z (w-z)^{j-1}\overline{(w-z)}^k = \frac{\partial p_2^z}{\partial w}(w),$$

and

$$\frac{\partial p}{\partial \bar{w}}(w) - i\frac{\partial T}{\partial \bar{w}}(w,z) = \sum_{j,k\geq 1} k a_{jk}^z (w-z)^{j}\overline{(w-z)}^{k-1} = \frac{\partial p_2^z}{\partial \bar{w}}(w).$$

We compute

$$\begin{aligned}
&U_z \mathcal{H}_{\tau p_1} U_z^{-1} f(s,w) \\
&= \frac{\partial f}{\partial s}(s,w) + U_z \Bigg[-\frac{\partial^2}{\partial w \partial \bar{w}}(U_z^{-1} f(s,w)) + \tau \frac{\partial^2 p_1}{\partial w \partial \bar{w}}(w) U_z^{-1} f(s,w) \\
&\quad + \tau^2 \left|\frac{\partial p_1}{\partial w}\right|^2 U_z^{-1} f(s,w) + \tau \left(\frac{\partial p_1}{\partial w}\frac{\partial}{\partial \bar{w}}(U_z^{-1} f(s,w)) - \frac{\partial p_1}{\partial \bar{w}}\frac{\partial}{\partial w}(U_z^{-1} f(s,w)) \right) \Bigg] \\
&= \frac{\partial f}{\partial s}(s,w) - \frac{\partial^2 f}{\partial w \partial \bar{w}}(s,w) + \tau^2 \frac{\partial T}{\partial w}(w,z)\frac{\partial T}{\partial \bar{w}}(w,z) f(s,w) + \tau i \frac{\partial T}{\partial w}(w,z)\frac{\partial f}{\partial \bar{w}}(s,w) \\
&\quad + \tau i \frac{\partial T}{\partial \bar{w}}(w,z)\frac{\partial f}{\partial w}(s,w) + \tau \frac{\partial^2 p_1}{\partial w \partial \bar{w}}(w) f(s,w) + \tau^2 \left|\frac{\partial p_1}{\partial w}\right|^2 f(s,w) + \tau i \frac{\partial p}{\partial w}(w,z)\frac{\partial f}{\partial \bar{w}}(s,w) \\
&\quad - \tau^2 i \frac{\partial p_1}{\partial w}\frac{\partial T}{\partial \bar{w}}(w,z) f(s,w) - \tau i \frac{\partial p}{\partial \bar{w}}(w,z)\frac{\partial f}{\partial w}(s,w) + \tau^2 i \frac{\partial p_1}{\partial \bar{w}}\frac{\partial T}{\partial w}(w,z) f(s,w) \\
&= \frac{\partial f}{\partial s}(s,w) - \frac{\partial^2 f}{\partial w \partial \bar{w}}(s,w) + \tau \frac{\partial^2 p_1}{\partial w \partial \bar{w}}(w) f(s,w) \\
&\quad + \tau^2 \left(\frac{\partial p_1}{\partial \bar{w}} - i\frac{\partial T}{\partial \bar{w}}(w,z) \right)\left(\frac{\partial p_1}{\partial w} + i\frac{\partial T}{\partial w}(w,z) \right) f(s,w) \\
&\quad + \tau \left(\left(\frac{\partial p_1}{\partial w} + i\frac{\partial T}{\partial w}(w,z) \right)\frac{\partial f}{\partial \bar{w}} - \left(\frac{\partial p_1}{\partial \bar{w}} - i\frac{\partial T}{\partial \bar{w}}(w,z) \right)\frac{\partial f}{\partial w} \right) \\
&= \mathcal{H}_{\tau p_2^z}[f](s,w).
\end{aligned}$$

Observe that

$$U_{z_0}^{-1}(\delta_0 \otimes \delta_z) U_{z_0} \psi(s,w) = U_{z_0}^{-1}(U_{z_0}\psi(0,z)) = \psi(0,z).$$

Using that $\mathcal{H}_{\tau p_2^{z_0}} U_{z_0} = U_{z_0}\mathcal{H}_{\tau p_1}$ implies $U_{z_0}\mathcal{H}^{\#}_{\tau p_2^{z_0}} = \mathcal{H}^{\#}_{\tau p_1} U_{z_0}$, we have

$$\begin{aligned}
U_z^{-1}(\mathcal{H}_{\tau p_1} H_z^{\tau p_1}\psi(s,\cdot))U_z^{-1}\psi &= U_z^{-1}\int_0^\infty\int_{\mathbb{C}}\Big(\frac{\partial}{\partial s}+\Box_{\tau p_1,z}\Big)H_{\tau p_1}(s,z,w)U_z\psi(s,w)\,dwds\\
&= U_z^{-1}\int_0^\infty\int_{\mathbb{C}}\Big(\frac{\partial}{\partial s}+\Box^{\#}_{\tau p_1,w}\Big)H_{\tau p_1}(s,z,w)U_z\psi(s,w)\,dwds\\
&= U_z^{-1}\int_0^\infty\int_{\mathbb{C}}H_{\tau p_1}(s,z,w)\Big(-\frac{\partial}{\partial s}+\Box_{\tau p_1,w}\Big)U_z\psi(s,w)\,dwds\\
&= \int_0^\infty\int_{\mathbb{C}}e^{-i\tau T(z,z_0)}H_{\tau p_1}(s,z,w)e^{i\tau T(w,z_0)}\Big(-\frac{\partial}{\partial s}+\Box_{\tau p_2,w}\Big)\psi(s,w)\,dwdw\\
&= \int_0^\infty\int_{\mathbb{C}}e^{-i\tau T(z,z_0)}\Big(\frac{\partial}{\partial s}+\Box^{\#}_{\tau p_2,w}\Big)\Big(e^{i\tau T(w,z_0)}H_{\tau p_1}(s,z,w)\Big)\psi(s,w)\,dwds.
\end{aligned}$$

Thus $H_{\tau p_2^{z_0}}(s,z_0,w) = e^{i\tau T(w,z_0)}H_{\tau p_1}(s,z_0,w)$. □

Let $T_\lambda\psi(s,w)=\lambda^2\psi(\lambda^2 s,\lambda w)$ and $T_\lambda f(w)=\lambda f(\lambda w)$ act on functions on $\mathbb{R}\times\mathbb{C}$ and $\mathbb{C}$, respectively. In either case, T_λ is an isometry on L^2. Our final proposition in this section investigates conjugating $\mathcal{H}_{\tau p}$ by T_λ. Let $\psi_\lambda(s,w)=\psi(\lambda^2 s,\lambda w)$ and $f_\lambda(w)=f(\lambda w)$.

Proposition 4.3. *If $p_3^\lambda = p_2(z/\lambda)$, then*

$$\frac{1}{\lambda^2}H_{\tau p_2^{z_0}}(s/\lambda^2, z/\lambda, w/\lambda) = H_{\tau p_3^\lambda}(s,z,w).$$

Proof.

$$\begin{aligned}
T_\lambda^{-1}\mathcal{H}_{\tau p}T_\lambda[f](s,w) &= \lambda^2 T_\lambda^{-1}\Bigg[\frac{\partial f_\lambda}{\partial s}(s,w)-\frac{\partial^2 f_\lambda}{\partial w\partial\bar w}(s,w)+\tau\frac{\partial^2 p}{\partial w\partial\bar w}(w)f_\lambda(s,w)\\
&\quad+\tau^2\left|\frac{\partial p}{\partial w}\right|^2 f_\lambda(s,w)+\tau\left(\frac{\partial p}{\partial w}(w)\frac{\partial f_\lambda}{\partial\bar w}(s,w)-\frac{\partial p}{\partial\bar w}(w)\frac{\partial f_\lambda}{\partial w}(s,w)\right)\Bigg]\\
&= \lambda^2 T_\lambda^{-1}\Bigg[\lambda^2\frac{\partial f}{\partial s}(\lambda^2 s,\lambda w)-\lambda^2\frac{\partial^2 f}{\partial w\partial\bar w}(\lambda^2 s,\lambda w)+\lambda^2\tau\frac{1}{\lambda^2}\frac{\partial^2 p}{\partial w\partial\bar w}(w)f(\lambda^2 s,\lambda w)\\
&\quad+\lambda^2\tau^2\left|\frac{1}{\lambda}\frac{\partial p}{\partial w}\right|^2 f(\lambda^2 s,\lambda w)+\lambda^2\tau\left(\frac{1}{\lambda}\frac{\partial p}{\partial w}(w)\frac{\partial f}{\partial\bar w}(\lambda^2 s,\lambda w)-\frac{1}{\lambda}\frac{\partial p}{\partial\bar w}(w)\frac{\partial f}{\partial w}(\lambda^2 s,\lambda w)\right)\Bigg]\\
&= \lambda^2\Bigg[\frac{\partial f}{\partial s}(s,w)-\frac{\partial^2 f}{\partial w\partial\bar w}(s,w)+\tau\frac{\partial^2 p_{\lambda^{-1}}}{\partial w\partial\bar w}(w)f(s,w)+\tau^2\left|\frac{\partial p_{\lambda^{-1}}}{\partial w}\right|^2 f(s,w)\\
&\quad+\tau\left(\frac{\partial p_{\lambda^{-1}}}{\partial w}(w)\frac{\partial f}{\partial\bar w}(s,w)-\frac{\partial p_{\lambda^{-1}}}{\partial\bar w}(w)\frac{\partial f_\lambda}{\partial w}(s,w)\right)\Bigg] = \lambda^2\mathcal{H}_{\tau p_{\lambda^{-1}}}[f](s,w).
\end{aligned}$$

Thus we have

$$T_\lambda^{-1}\mathcal{H}_{\tau p_2} = \lambda^2 \mathcal{H}_{\tau p_3^\lambda} T_\lambda^{-1}.$$

Next,

$$T_\lambda^{-1}(\delta_0 \otimes \delta_z)T_\lambda[f] = \lambda^2 T_\lambda^{-1} f(0, \lambda z) = f(0, z),$$

so

$$\begin{aligned}
T_\lambda^{-1}\delta_0 \otimes \delta_z T_\lambda[f] &= \lambda^2 T_\lambda^{-1}\mathcal{H}_{\tau p_2} \int_0^\infty \int_{\mathbb{C}} H_{\tau p_2}(s, z, w) f(\lambda^2 s, \lambda w)\, dwds \\
&= \lambda^2 T_\lambda^{-1} \int_0^\infty \int_{\mathbb{C}} \left(\frac{\partial}{\partial s} + \Box^\#_{\tau p_2}\right) H_{\tau p_2}(s, z, w) f(\lambda^2 s, \lambda w)\, dwds \\
&= T_\lambda^{-1} \int_0^\infty \int_{\mathbb{C}} T_\lambda^{-1}\Big(\frac{\partial}{\partial s} + \Box^\#_{\tau p_2}\Big) H_{\tau p_2}(s, z, w) f(\lambda^2 s, \lambda w)\, dwds \\
&= T_\lambda^{-1} \int_0^\infty \int_{\mathbb{C}} T_\lambda^{-1} \left(\frac{\partial}{\partial s} + \Box^\#_{\tau p_2}\right) H_{\tau p_2}(s, z, w) f(s, w)\, dwds \\
&= \lambda^2 T_\lambda^{-1} \int_0^\infty \int_{\mathbb{C}} \left(\frac{\partial}{\partial s} + \Box^\#_{\tau p_3^\lambda}\right) \Big(T_\lambda^{-1} H_{\tau p_2}(s, z, w)\Big) f(s, w)\, dwds \\
&= \int_0^\infty \int_{\mathbb{C}} \left(\frac{\partial}{\partial s} + \Box^\#_{\tau p_3^\lambda}\right) \frac{1}{\lambda^2} H_{\tau p_2}(s/\lambda^2, z/\lambda, w/\lambda) f(s, w)\, dwds.
\end{aligned}$$

The conclusion follows immediately. □

4.2 Pointwise Estimates for $|H_{\tau p}(s, z, w)|$

We first show that $|H_{\tau p}(s, z, w)|$ has Gaussian decay. To do so, we will find it convenient to work in real variable notation instead of complex notation. As such, let $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Our first goal is to prove:

Theorem 4.4. *If $e^{-s\Box_{\tau p}}[f](x) = \int_{\mathbb{C}} H_{\tau p}(s, x, y) f(y)\, dy$, the heat kernel $H_{\tau p}(s, x, y)$, satisfies the estimate*

$$|H_{\tau p}(s, x, y)| \le \frac{1}{\pi s} e^{-\frac{|x-y|^2}{s}}.$$

Proof. We will use the Feynman-Kac-Itô formula from [Sim79]. Let dx be Lebesgue measure on $\mathbb{R}^2$ and let $(B, \mathfrak{B}, dP)$ be a measure space of sample paths for a 2-dimensional Brownian motion $b(s)$. Let $d\mu = dP \otimes dx$ be Wiener measure on $B \times \mathbb{R}^2$ and let $\omega(s) = x + b(s)$. If we let

$$a(x) = \tau \left(-\frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_1} \right)$$

and

$$V(x) = \frac{\tau}{2} \triangle p(x),$$

then for $f \in C^2(\mathbb{R}^2)$,

$$\frac{1}{2}(-i\nabla - a)^2 f + Vf = -\frac{1}{2}\triangle f + \frac{i}{2}(\nabla \cdot a)f + ia \cdot \nabla f + \frac{1}{2}|a|^2 f + Vf.$$

But $\nabla \cdot a = \tau(-\frac{\partial^2 p}{\partial x_1 \partial x_2} + \frac{\partial^2 p}{\partial x_1 \partial x_2}) = 0$ and $\frac{1}{2}|a|^2 = \frac{1}{2}\tau^2 |\nabla p|^2$. Thus,

$$\frac{1}{2}(-i\nabla - a)^2 f + Vf = 2\Box_{\tau p},$$

so $2\Box_{\tau p}$ is the quantum mechanical energy operator for a particle in a magnetic field with vector potential $a(x)$ and electric potential V. The Feynman-Kac-Itô formula for for $f, g \in C_c^\infty(\mathbb{R}^2)$ is

$$\left(e^{-2s\Box_{\tau p}} f, g \right) = \int e^{F(s,\omega)} f(\omega(s)) \overline{g(\omega(0))} \, d\mu \tag{4.3}$$

where

$$F(s, \omega) = -i \int_0^s a(\omega(t)) \cdot d\omega(t) - \frac{i}{2} \int_0^s (\nabla \cdot a)(\omega(t)) \, dt - \int_0^s V(\omega(t)) \, dt.$$

$b(s)$ has 2-dimensional normal distribution with covariance s, so we can rewrite (4.3) as follows:

$$\begin{aligned}
\iint_{\mathbb{R}^2\times\mathbb{R}^2} H_{\tau p}(2s,x,y)f(y)\overline{g(x)}\,dydx &= \int e^{F(s,\omega)}f(\omega(s))\overline{g(\omega(0))}\,d\mu\\
&= \int_{\mathbb{R}^2} E\big[e^{F(s,\omega)}f(\omega(s))\overline{g(\omega(0))}\big]\,dx\\
&= \int_{\mathbb{R}^2} E\big[E[e^{F(s,\omega)}f(\omega(s))\overline{g(\omega(0))}\big|\omega(0)=x,\omega(s)=y]\big]\,dx\\
&= \int_{\mathbb{R}^2} E\big[E[e^{F(s,\omega)}\big|\omega(0)=x,\omega(s)=y]\big]f(y)\overline{g(x)}\,dx\\
&= \frac{1}{2\pi s}\iint_{\mathbb{R}^2\times\mathbb{R}^2} e^{\tilde{F}(s,x+y)}f(x+y)\overline{g(x)}e^{-\frac{|y|^2}{2s}}\,dydx\\
&= \frac{1}{2\pi s}\iint_{\mathbb{R}^2\times\mathbb{R}^2} e^{\tilde{F}(s,y)}f(y)\overline{g(x)}e^{-\frac{|x-y|^2}{2s}}\,dydx.
\end{aligned}$$

Thus, $H_{\tau p}(2s,x,y) = e^{\tilde{F}(s,y)}e^{-\frac{|x-y|^2}{2s}}$ for some $\tilde{F}(s,y)$ satisfying $|e^{\tilde{F}(s,y)}| \leq 1$. That $|e^{\tilde{F}(s,y)}| \leq 1$ follows from $|e^{F(s,\omega)}| \leq 1$. □

A critically important fact about the Feynman-Kac-Itô formula is the requirement that $V \geq 0$. When $\tau < 0$, $V \leq 0$, and the argument from Theorem 4.4 fails. Even if we could use the argument, the real part of $e^{F(s,\omega)}$ is $e^{-\int_0^s V(\omega(t))\,dt}$, a term that we would expect to be very large. Qualitatively, Feynman-Kac-Itô is the wrong direction to push. In fact, since it is equivalent to study the $\tau < 0$ and $\Box_{\tau p}$ or $\tau > 0$ and $\widetilde{\Box}_{\tau p}$, an analog to Theorem 4.4 must fail if $\tau < 0$ because $\lim_{s\to\infty} \tilde{H}_{\tau p}(s,z,w) = S_{\tau p}(z,w)$.

We now turn to proving a large time decay estimate for $H_{\tau p}(s,z,w)$. Let P^{2m} be the set of degree $2m$ polynomials whose coefficients (in absolute value) sum to 1. We can identify the set of polynomials of degree $2m$ with $\mathbb{R}^n$ for some n, and under this identification, P^{2m} is identified with the unit sphere, a compact set. Having constants depending only on P^{2m} is essential for estimates obtained through scaling.

Theorem 4.5. *If $e^{-s\Box_{\tau p}}[f](z) = \int_{\mathbb{C}} H_{\tau p}(s,z,w)f(w)\,dw$, then there exist constants C_1*

and C_2 which depend on the degree of p so that

$$|H_{\tau p}(s,z,w)| \le \frac{C_1}{s} e^{-C_2 \frac{s}{\mu(z,\frac{1}{\tau})^2}}.$$

Proof. From Proposition 4.1, there exists a polynomial $p_1(w) = p_0(w+z)$ so that

$$H_{\tau p}(s,z,w) = H_{\tau p_1}(s,0,w-z),$$

so we can reduce to the case of estimating $H_{\tau p_1}(s,0,w)$. By Proposition 4.2, for

$$p_2(w) = \sum_{j,k\ge 1} a^0_{jk} w^j \bar{w}^k,$$

we have

$$e^{i\tau T(w,0)} H_{\tau p_1}(s,0,w) = H_{\tau p_2}(s,0,w),$$

so it is enough to estimate $|H_{\tau p_2}(s,0,w)|$. $p_2(w)$ has the property that $\frac{\partial^k p_2}{\partial z^k}(0) = \frac{\partial^k p_2}{\partial \bar{z}^k}(0) = 0$ for all k. If we set $\lambda = \mu(z,\frac{1}{\tau})^{-1}$ and $p_3(w) = p_2(\frac{w}{\lambda})$, then $p_3 \in P^{2m}$ since

$$\tau \sum_{j,k\ge 1} \frac{1}{j!k!} \left| \frac{\partial^{j+k} p_2}{\partial z^j \partial \bar{z}^k}(0) \right| z^j \bar{z}^k \lambda^{-j-k} = \tau \Lambda\big(0, \mu(0,\tfrac{1}{\tau})\big) \sim 1.$$

From Proposition 4.3,

$$\frac{1}{\lambda^2} H_{\tau p_2}(\tfrac{s}{\lambda^2}, 0, \tfrac{w}{\lambda}) = H_{p_3}(s,0,w).$$

We now estimate $|H_{p_3}(s,w,0)|$. Let $h(s,w) = H_{p_3}(s,w,0)$. Then $\frac{\partial h}{\partial s} - \overline{Z}_{p_3} Z_{p_3} h = 0$, so

$$\frac{\partial h}{\partial s} = \overline{Z}_{p_3} Z_{p_3} h.$$

Let $g(s) = \int_{\mathbb{C}} |h(s,w)|^2\, dw$. From [Chr91a], $\|f\|_{L^2(\mathbb{C})} \le C \|Z_{p_3} f\|_{L^2(\mathbb{C})}$ where $C =$

$C(P^{2m})$, so

$$\begin{aligned} g'(s) &= \int_{\mathbb{C}} \frac{d}{ds}\big(h(s,w)\overline{h(s,w)}\big)\,dw \\ &= 2\,\mathrm{Re}\int_{\mathbb{C}} \frac{\partial h}{\partial s}(s,w)\overline{h(s,w)}\,dw \\ &= 2\,\mathrm{Re}\int_{\mathbb{C}} \overline{Z}_{p_3} Z_{p_3} h(s,w)\overline{h(s,w)}\,dw \\ &= -2\int_{\mathbb{C}} \big|Z_{p_3}h(s,w)\big|^2\,dw \\ &\le -C\int_{\mathbb{C}} \big|h(s,w)\big|^2\,dw = -Cg(s). \end{aligned}$$

Since $g(s) > 0$, $\frac{g'(s)}{g(s)} \le -C$, and integrating from $\frac{s}{2}$ to s, we have

$$g(s) \le g(\tfrac{s}{2})e^{-Cs} \le C_1\frac{e^{-Cs}}{s},$$

where the last inequality follows from Theorem 4.4. The constant C_1 does not depend on p_3 (or P^{2m}).

Next, $e^{-s\Box_{p_3}}$ is a semigroup, so $e^{-\frac{s}{2}\Box_{p_3}}e^{-\frac{s}{2}\Box_{p_3}}f(z) = e^{-s\Box_{p_3}}f(z)$.

$$e^{-s\Box_{p_3}}f(z) = \int_{\mathbb{C}} H_{p_3}(s,z,w)f(w)\,dw,$$

and

$$e^{-\frac{s}{2}\Box_{p_3}}e^{-\frac{s}{2}\Box_{p_3}}f(z) = \int_{\mathbb{C}} H_{p_3}(\tfrac{s}{2},z,v)e^{-\frac{s}{2}\Box_{p_3}}f(v)\,dv = \int_{\mathbb{C}}\int_{\mathbb{C}} H_{p_3}(\tfrac{s}{2},z,v)H_{p_3}(\tfrac{s}{2},v,w)f(w)\,dwdv.$$

Thus we have the reproducing identity

$$H_{p_3}(s,z,w) = \int_{\mathbb{C}} H_{p_3}(\tfrac{s}{2},z,v)H_{p_3}(\tfrac{s}{2},v,w)\,dv,$$

and an application of Cauchy-Schwarz yields

$$\begin{aligned} \big|H_{p_3}(2s,0,w)\big| &\le \left(\int_{\mathbb{C}} \big|H_{p_3}(s,0,v)\big|^2\,dv\right)^{\frac{1}{2}}\left(\int_{\mathbb{C}} \big|H_{p_3}(s,v,w)\big|^2\,dv\right)^{\frac{1}{2}} \\ &\le C_1\frac{e^{-Cs}}{s}. \end{aligned}$$

Undoing the rescaling shows $|H_{\tau p_2}(\mu(0,\frac{1}{\tau})^2 s, 0, \mu(0,\frac{1}{\tau})w)| \leq \frac{C_1}{s}\frac{1}{\mu(0,\frac{1}{\tau})^2}e^{-C_2 s}$, so

$$|H_{\tau p_2}(s,0,w)| \leq \frac{C_1}{s}e^{-C_2\frac{s}{\mu(0,1/\tau)^2}},$$

and

$$\left|H_{\tau p}(s,z,w)\right| \leq \frac{C_1}{s}e^{-C_2\frac{s}{\mu(z,1/\tau)^2}}.$$

□

The motivation for using $g'(s)$ and the reproducing identity was [Fab93].

4.3 Derivative Estimates

The derivative estimates are proven in a series of lemmas. The most accessible case is proven first and each successive lemma builds on the previous calculation. Each L^2 estimate at one step is used to prove a pointwise estimate in the next. Define the decay term $D(s,x,y)$ to be

$$D(s,x,y) = e^{-\frac{|x-y|^2}{2s}}e^{-C_2\frac{2}{\mu(x,1/\tau)^2}} \tag{4.4}$$

where C_2 is the constant from Theorem 4.5. Also, let

$$I_r(s) = (s-r^2, s) \text{ and } Q_r(s,x) = I_r(s)\times D(x,r).$$

Proposition 4.6. *There exists C_n so that for $0 < r < \frac{\sqrt{s_0}}{16}$,*

$$\left\|\frac{\partial^n H_{\tau p}}{\partial s^n}(\cdot,x,\cdot)\right\|_{L^2(Q_r(s_0,y_0))} \leq \frac{C}{s_0^n}.$$

Proof. We have

$$\left\|\frac{\partial^n H_{\tau p}}{\partial s^n}(\cdot,x_0,\cdot)\right\|^2_{L^2(Q_r(s_0,y_0))} = \int_{I_r(s_0)}\left|\left(\int_{D(y_0,r)}\left|\frac{\partial^n H_{\tau p}}{\partial s^n}(s,x_0,y)\right|^2 dy\right)^{1/2}\right|^2 ds$$
$$= \int_{I_r(s_0)}\left|\sup_{\substack{\varphi\in C_c^\infty(D(y_0,r))\\ \|\varphi\|_{L^2}=1}}\int \frac{\partial^n H_{\tau p}}{\partial s^n}(s,x_0,y)\varphi(y)\,dy\right|^2 ds$$
$$= \int_{I_r(s_0)}\left|\sup_{\substack{\varphi\in C_c^\infty(D(y_0,r))\\ \|\varphi\|_{L^2}=1}}\frac{\partial^n}{\partial s^n}H^s_{\tau p}[\varphi](x_0)\right|^2 ds. \tag{4.5}$$

The key to the proof is that $\frac{\partial^n}{\partial s^n}H^s_{\tau p}[\varphi](x)$ satisfies $(\frac{\partial}{\partial s}+\Box_x)\frac{\partial^n}{\partial s^n}H_{\tau p}(s,x,y) = 0$. By Lemma A.1 and Theorem 3.1 (d), estimating an arbitrary term from the supremum in (4.5) yields

$$\left|\frac{\partial^n}{\partial s^n}H^s_{\tau p}[\varphi](x_0)\right| \le \frac{C}{r^2}\left(\iint_{Q_r(s,x_0)}\left|\frac{\partial^n}{\partial s^n}H^t_{\tau p}[\varphi](x)\right|^2 dxdt\right)^{1/2}$$
$$\le \frac{C}{r^2}\left(\int_{I_{\sqrt{2}r}(s_0)}\left\|\frac{\partial^n}{\partial s^n}H^t_{\tau p}[\varphi]\right\|^2_{L^2} dt\right)^{1/2}$$
$$\le \frac{C}{r^2}\left(\int_{s_0-2r^2}^{s_0}\frac{1}{t^{2n}}\,dt\right)^{1/2} \le \frac{C}{rs_0^n} \tag{4.6}$$

Putting (4.5) into (4.6), we have

$$\left\|\frac{\partial^n H_{\tau p}}{\partial s^n}(\cdot,x_0,\cdot)\right\|_{L^2(Q_r(s_0,y_0)} \le C\left(\int_{I_r(s_0)}\frac{1}{r^2 s_0^{2n}}\,ds\right)^{1/2} = \frac{C}{s_0^n}.$$

□

Lemma 4.7. *Let n_1, n_2, $n_3 \ge 0$ and $n = n_1+n_2+n_3$. Then there exists $C_n > 0$ so that*

$$\left|\frac{\partial^{n_1}}{\partial s^{n_1}}\Box^{n_2}_{\tau p,x}(\Box^{\#}_{\tau p,y})^{n_3}H_{\tau p}(s,x,y)\right| \le \frac{C_n}{s_0^{n+1}}D(s,x,y)^{\frac12}$$

Proof. Since $H_{\tau p}$ satisfies $(\frac{\partial}{\partial s}+\Box_{\tau p,x})H_{\tau p}(s,x,y) = 0$ when $s\ne 0$ or $x\ne y$, it is enough to show the estimate for $H_n(s,x,y) = \frac{\partial^n}{\partial s^n}H_{\tau p}(s,x,y)$. Proof by induction. The base case follows from combining Theorem 4.4 and Theorem 4.5.

$$|H_{\tau p}(s,x,y)| \leq |H_{\tau p}(s,x,y)|^{\frac{1}{2}} |H_{\tau p}(s,x,y)|^{\frac{1}{2}} \leq \frac{C}{s} e^{-\frac{|x-y|^2}{2s}} e^{-C_2 \frac{s}{\mu(x,1/\tau)^2}}.$$

Assume the result holds for H_{n-1}. Let $r = \frac{\sqrt{s_0}}{16}$. Let $\psi \in C_c^\infty\big(Q_{2r}(s_0, y_0)\big)$ where $\psi\Big|_{Q_r(s_0,y_0)} \equiv 1$, $0 \leq \psi \leq 1$, and $\frac{\partial^j \psi}{\partial s^j} \leq \frac{c_j}{r^{2j}}$. We can use Lemma A.1 because if $s > 0$, $H_{n-1}(s,z,w)$ satisfies $\mathcal{H}_{\tau p} H_{n-1}(s,x,y) = 0$. Using Lemma A.1 and Proposition 4.6, for $r > 0$ and $Q = Q_{2r}(s_0, y_0)$

$$\begin{aligned}
&\left| \frac{\partial^n H_{\tau p}}{\partial s^n}(s_0, x, y_0) \right| \leq \frac{C}{r^2} \left(\iint_{Q_r(s_0,y_0)} \left| \frac{\partial^n H_{\tau p}}{\partial s^n}(s,x,y) \right|^2 dsdy \right)^{\frac{1}{2}} \\
&\leq \frac{C}{r^2} \left(\iint_{\mathbb{R}\times\mathbb{C}} \frac{\partial^n H_{\tau p}}{\partial s^n}(s,x,y) \overline{\frac{\partial^n H_{\tau p}}{\partial s^n}(s,x,y)} \psi(s,y)\, dsdy \right)^{\frac{1}{2}} \\
&= \frac{C}{r^2} \left(\iint_{\mathbb{R}\times\mathbb{C}} \overline{H_{\tau p}(s,x,y)} \sum_{j=0}^{n} \frac{\partial^{n+j} H_{\tau p}}{\partial s^{n+j}}(s,x,y) \frac{\partial^{n-j}\psi}{\partial s^{n-j}}(s,y)\, dsdy \right)^{\frac{1}{2}} \\
&\leq \frac{C}{r^2} \left[\|H_{\tau p}(\cdot, x, \cdot)\|_{L^2(Q)} \sum_{j=0}^{n} c_j \frac{1}{r^{2(n-j)}} \left\| \frac{\partial^{n+j} H_{\tau p}}{\partial s^{n+j}}(\cdot, x, \cdot) \right\|_{L^2(Q)} \right]^{\frac{1}{2}} \\
&\leq \frac{C}{r^2} \left[H_{\tau p}(s_0, x, y_0) r^2 \left(\frac{1}{s_0^{2n}} + \frac{1}{r^{2n} s_0^n} \right) \right]^{1/2} \\
&\leq \frac{C_n}{r} \frac{D(s_0,x,y_0)^{\frac{1}{2}}}{s_0^{\frac{1}{2}}} \left(\frac{1}{s_0^n} + \frac{1}{r^n s_0^{\frac{n}{2}}} \right) \leq \frac{C_n}{s_0^{n+1}} D(s_0,x,y_0)^{\frac{1}{2}}
\end{aligned}$$

□

Integrating in s gives the immediate corollary:

Corollary 4.8. *Let n_1, n_2, $n_3 \geq 0$ and $n = n_1 + n_2 + n_3$. Then there exists $C_n > 0$ so that*

$$\left\| \frac{\partial^{n_1}}{\partial s^{n_1}} \Box_{\tau p,x}^{n_2} (\Box_{\tau p,y}^{\#})^{n_3} H_{\tau p}(s,x,y) \right\|_{L^2(\mathbb{C})} \leq \frac{C_n}{s^{n+\frac{1}{2}}}.$$

□

Lemma 4.9. *Let α be a multiindex and $j \geq 0$. Then there exists $C_{|\alpha|,j} > 0$ so that if $R = \min\{\frac{\sqrt{s_0}}{16}, \frac{\mu(x_0,\frac{1}{\tau})}{4}\}$, then*

$$\left|X_x^\alpha(\Box_{\tau p,y}^{\#})^j H_{\tau p}(s,x,y)\right| + \left|\Box_{\tau p,x}^j U_y^\alpha H_{\tau p}(s,x,y)\right| \leq \frac{C_{|\alpha|}}{R^{\frac{1}{2}} s^{\frac{3}{4}+j}} D(s,x_0,y)^{\frac{1}{4}} R^{-\frac{1}{2}|\alpha|} s^{-\frac{1}{4}|\alpha|}.$$

Proof. It is enough to bound $|U_y^\alpha \Box_{\tau p,x}^j H_{\tau p}(s,x_0,y)|$ for a fixed $x_0 \in \mathbb{C}$. In fact, we can even assume that $\frac{\partial^n p}{\partial z^n}(x_0) = \frac{\partial^n p}{\partial \bar{z}^n}(x_0) = 0$ for all n by Proposition 4.2. This means if $|y - x_0| \leq \mu(x_0, \frac{1}{\tau})$,

$$\left|\frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(y)\right| \leq \frac{1}{\mu(x_0,\frac{1}{\tau})^{j+k}} \tau \Lambda(x_0, \mu(x_0,\tfrac{1}{\tau})) \sim \frac{1}{\mu(x_0,\frac{1}{\tau})^{j+k}} \tag{4.7}$$

Let $R = \min\{\frac{\sqrt{s}}{16}, \frac{1}{4}\mu(x_0,\frac{1}{\tau})\}$. Also, fix s and let $g(y) = \Box_{\tau p,x}^j H_{\tau p}(s,x_0,y)$. Let D stand for $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$ Then from Thereom 3.10, if $\varphi \in C_c^\infty(D(x_0,R))$ with $0 \leq \varphi \leq 1$, $|D^\beta \varphi| \leq \frac{c_{|\beta|}}{R^{|\beta|}}$ for $0 \leq |\beta| \leq 2$, we have

$$|U_y^\alpha g(y)| \leq \frac{C}{R} \sum_{|\beta|\leq 2} R^{|\beta|} \|\varphi^{\frac{1}{2}} U_y^\beta U_y^\alpha g\|_{L^2(\mathbb{C})}. \tag{4.8}$$

Then

$$\begin{aligned}
\|\varphi^{\frac{1}{2}} U_y^\beta U_y^\alpha g\|_{L^2(\mathbb{C})}^2 &\leq \Big(U_y^\beta U_y^\alpha g, \varphi U_y^\beta U_y^\alpha g\Big) \\
&= \Big|\Big(g, U_y^\beta U_y^\alpha\big(\varphi U_y^\beta U_y^\alpha g\big)\Big)\Big| \\
&= \sum_{|\gamma_1|+|\gamma_2|=|\alpha|+|\beta|} c_{\gamma_1,\gamma_2}\Big(g, (D^{\gamma_1}\varphi) U_y^{\gamma_2} U_y^\beta U_y^\alpha g\Big) \\
&\leq \sum_{|\gamma_1|+|\gamma_2|=|\alpha|+|\beta|} c_{\gamma_1,\gamma_2} \|g\|_{L^2(D(x_0,R))} \frac{1}{R^{|\gamma_1|}} \|U_y^{\gamma_2} U_y^\beta U_y^\alpha g\|_{L^2(D(x_0,R))}
\end{aligned} \tag{4.9}$$

Next, from Corollary 4.8, Proposition 3.7, and Theorem 3.1, for some OPF operator of order 0 B_τ, we have the estimate (note the complex conjugate in the first inequality),

$$\begin{aligned}
\|U_y^{\gamma_2}U_y^{\beta}U_y^{\alpha}g\|^2_{L^2(D(x_0,R))} &\le \left(X_y^{\gamma_2}X_y^{\beta}X_y^{\alpha}\bar{g}, X_y^{\gamma_2}X_y^{\beta}X_y^{\alpha}\bar{g}\right)\\
&= \left(\bar{g}, X_y^{\alpha}X_y^{\beta}X_y^{\gamma_2}X_y^{\gamma_2}X_y^{\beta}X_y^{\alpha}\bar{g}\right)\\
&\le \|g\|_{L^2(\mathbb{C})}\|B_\tau\Box_{\tau p}^{|\gamma_2|+|\beta|+|\alpha|}\bar{g}\|_{L^2(\mathbb{C})}\\
&\le \frac{C}{s^{\frac{1}{2}+j}}s^{-(|\gamma_2|+|\beta|+|\alpha|+\frac{1}{2})}. \qquad (4.10)
\end{aligned}$$

Plugging (4.9) into (4.10) gives

$$\|\varphi^{\frac{1}{2}}X_y^{\beta}X_y^{\alpha}\bar{g}\|^2_{L^2(\mathbb{C})} \le C|g(y)|R\sum_{|\gamma_1|+|\gamma_2|=|\alpha|+|\beta|}c_{\gamma_1,\gamma_2}\frac{1}{R^{|\gamma_1|}}s^{-\frac{1}{2}(|\gamma_2|+|\beta|+|\alpha|+1+2j)} \qquad (4.11)$$

Using the fact that $R \le \sqrt{s}$ and inserting (4.11) into (4.8), we have

$$\begin{aligned}
|X_y^{\alpha}\overline{g(y)}| &\le \frac{C_{|\alpha|}}{R}\sum_{|\beta|\le 2}R^{|\beta|}|g(y)|^{\frac{1}{2}}R^{\frac{1}{2}}\sum_{|\gamma_1|+|\gamma_2|=|\alpha|+|\beta|}s^{-\frac{1}{4}(|\gamma_2|+|\beta|+|\alpha|+1+2j)}\frac{1}{R^{\frac{|\gamma_1|}{2}}}\\
&\le \frac{C_{|\alpha|}}{R^{\frac{1}{2}}s^{\frac{3}{4}+j}}D(s,x_0,y)^{\frac{1}{4}}\sum_{|\beta|\le 2}\sum_{|\gamma_1|+|\gamma_2|=|\alpha|+|\beta|}R^{|\beta|-\frac{1}{2}|\gamma_1|}s^{-\frac{1}{4}|\gamma_2|}s^{-\frac{1}{4}|\beta|}s^{-\frac{1}{4}|\alpha|}\\
&\le \frac{C_{|\alpha|}}{R^{\frac{1}{2}}s^{\frac{3}{4}+j}}D(s,x_0,y)^{\frac{1}{4}}R^{-\frac{1}{2}|\alpha|}s^{-\frac{1}{4}|\alpha|}.
\end{aligned}$$

□

Corollary 4.10. *Let α be a multiindex and $j \ge 0$. Then there exists $C_{|\alpha|,j} > 0$ so that*

$$\left\|X_x^{\alpha}(\Box^{\#}_{\tau p,y})^j H_{\tau p}(s,x,\cdot)\right\|_{L^2(\mathbb{C})} + \left\|U_y^{\alpha}\Box^j_{\tau p,x}H_{\tau p}(s,x,\cdot)\right\|_{L^2(\mathbb{C})} \le \frac{C_{|\alpha|,j}}{s^{\frac{1}{2}+j+\frac{|\alpha|}{2}}}.$$

Proof. Using the estimate from Lemma 4.9, if $R = \frac{\sqrt{s}}{16}$, then the result follows by direct calculation and a simple change of variables. If $R = \frac{1}{4}\mu(x,\frac{1}{\tau})$, then we use the fact that $D(s,x,y)^{\frac{1}{4}} \le C_j D(s,x,y)^{\frac{1}{8}}\left(\frac{\mu(x,\frac{1}{\tau})^2}{s}\right)^j$ for any $j \ge 0$. With this estimate, the result follows immediately. □

The final lemma we need is:

Lemma 4.11. *Let α and β be multiindices. There exists $C_{|\alpha|,|\beta|} > 0$ so that if $R = \min\{\frac{\sqrt{s_0}}{16}, \frac{\mu(x_0,\frac{1}{\tau})}{4}\}$, then*

$$\left|X_x^\alpha X_y^\beta H_{\tau p}(s,x,y)\right| \le C_{|\alpha|,|\beta|} \frac{1}{R^{\frac{3}{4}} s^{\frac{5}{8}}} R^{-\frac{|\alpha|}{2}-\frac{|\beta|}{4}} s^{-\frac{|\alpha|}{4}-\frac{3|\beta|}{8}} D(s,x,y)^{\frac{1}{2}}.$$

Proof. As in Lemma 4.9, we may assume that $\frac{\partial^n p}{\partial z^n}(x_0) = \frac{\partial^n p}{\partial \bar{z}^n}(x_0) = 0$ for all n so by (4.7)

$$\left|\frac{\partial^{j+k} p}{\partial z^j \partial \bar{z}^k}(y)\right| \lesssim \frac{1}{\mu(x_0,\frac{1}{\tau})^{j+k}}.$$

Fix s and x_0. Let $y \in D(x_0,R)$. Let $\varphi \in C_c^\infty(D(x_0,2R))$ so that $\varphi\Big|_{D(x_0,R)} \equiv 1$, $0 \le \varphi \le 1$ and $|D^\alpha \varphi| \le \frac{c_{|\alpha|}}{R^{|\alpha|}}$. Let $f(x) = X_y^\beta H_{\tau p}(s,x,y)$ and $g(x) = X_x^\alpha X_y^\beta H_{\tau p}(s,x,y)$. From Theorem 3.10,

$$|g(x_0)| \le \frac{C}{R} \sum_{|\gamma|\le 2} R^{|\gamma|} \|\varphi^{\frac{1}{2}} X_x^\gamma g\|_{L^2(\mathbb{C})}.$$

Next,

$$\begin{aligned}
\|\varphi^{\frac{1}{2}} X_x^\gamma g\|^2_{L^2(\mathbb{C})} &= \Big(X_x^\gamma X_x^\alpha f, \varphi X_x^\gamma g\Big) = \Big|\Big(f, X_x^\alpha X_x^\gamma\big[\varphi X_x^\gamma g\big]\Big)\Big| \\
&= \sum_{|\gamma_1|+|\gamma_2|=|\gamma|+|\alpha|} c_{\gamma_1,\gamma_2} \Big|\Big(f, D^{\gamma_1}\varphi X_x^{\gamma_2} X_x^\gamma g\Big)\Big| \\
&\le \sum_{|\gamma_1|+|\gamma_2|=|\gamma|+|\alpha|} c_{\gamma_1,\gamma_2} \|f\|_{L^2(D(x_0,R))} \frac{1}{R^{|\gamma_1|}} \|X_x^{\gamma_2} X_x^\gamma g\|_{L^2(\mathbb{C})}.
\end{aligned}$$

Using Proposition 3.7 and Corollary 4.10, for some order zero OPF operator B_τ we have

$$\begin{aligned}
\|X_x^{\gamma_2} X_x^\gamma g\|^2_{L^2(\mathbb{C})} &= \Big(X_x^{\gamma_2} X_x^\gamma X_x^\alpha f, X_x^{\gamma_2} X_x^\gamma X_x^\alpha f\Big) \\
&= \Big|\Big(f, X_x^\alpha X_x^\gamma X_x^{\gamma_2} X_x^{\gamma_2} X_x^\gamma X_x^\alpha f\Big)\Big| \\
&\le \|f\|_{L^2(\mathbb{C})} \|B_\tau \Box_{\tau p}^{|\alpha|+|\gamma|+|\gamma_2|} f\|_{L^2(\mathbb{C})} \\
&\le C_{|\alpha|+|\gamma|+|\gamma_2|+|\beta|} s^{-\frac{1}{2}(|\beta|+1)} s^{-\frac{1}{2}(|\beta|+1)-|\alpha|-|\gamma|-|\gamma_2|} \\
&= C_{|\alpha|+|\gamma|+|\gamma_2|+|\beta|} s^{-|\beta|-1-|\alpha|-|\gamma|-|\gamma_2|}.
\end{aligned}$$

Thus, since $\|f\|_{L^2(D(x_0,R))} \le C|f(x_0)|R$,

$$|g(x_0)| \le C_{|\alpha|,|\beta|}\frac{1}{R}\sum_{|\gamma|\le 2}\sum_{|\gamma_1|+|\gamma_2|=|\gamma|+|\alpha|} R^{|\gamma|}R^{-\frac{|\gamma_1|}{2}}s^{-\frac{1}{4}(|\beta|+1+|\alpha|+|\gamma|+|\gamma_2|)}|f(x_0)|^{\frac{1}{2}}R^{\frac{1}{2}}$$

$$\le C_{|\alpha|,|\beta|}\frac{1}{R^{\frac{3}{4}}s^{\frac{5}{8}}}\sum_{|\gamma|\le 2}\sum_{|\gamma_1|+|\gamma_2|=|\gamma|+|\alpha|} R^{|\gamma|}R^{-\frac{|\gamma_1|}{2}}s^{-\frac{|\gamma_2|}{4}}s^{-\frac{1}{4}(|\gamma|+|\alpha|)}s^{-\frac{|\beta|}{4}}s^{-\frac{|\beta|}{8}}R^{-\frac{|\beta|}{4}}D(s,x,y)^{\frac{1}{8}}$$

$$\le C_{|\alpha|,|\beta|}\frac{1}{R^{\frac{3}{4}}s^{\frac{5}{8}}}R^{-\frac{|\alpha|}{2}}s^{-\frac{|\alpha|}{4}}R^{-\frac{|\beta|}{4}}s^{-\frac{3}{8}|\beta|}$$

□

As a consequence of Lemma 4.11, we have:

Theorem 1.3. *Let $n \ge 0$ and Y^α be a product of $|\alpha|$ operators $Y = \bar{Z}_{\tau p}$ or $Z_{\tau p}$ if acting in z and $\overline{(Z_{\tau p})}$ or $\overline{(\bar{Z}_{\tau p})}$ if acting in w. There exists constants $c_1, c_2, c_3 > 0$ so that if $\tau > 0$,*

$$\left|\frac{\partial^n}{\partial s^n}Y^\alpha H_{\tau p}(s,z,w)\right| \le c_1\frac{1}{s^{n+\frac{1}{2}|\alpha|+1}}e^{-c_2\frac{|z-w|^2}{s}}e^{-c_3\frac{s}{\mu(z,\frac{1}{\tau})^2}}.$$

Proof. The theorem follows Lemma 4.11 using the argument of the proof of Corollary 4.10. □

Using Theorem 1.3, we can integrate in s and recover estimates on $G_{\tau p}(z,w)$, the fundamental solution of $\Box_{\tau p}$.

Corollary 1.4. *Let $G_{\tau p}(z,w)$ be the fundamental solution for $\Box_{\tau p}^{-1}$. There exists constants $C_1, C_2 > 0$ so that if $\tau > 0$,*

$$|G_{\tau p}(z,w)| \le C_1\begin{cases}\log\left(\frac{2\mu(z,\frac{1}{\tau})}{|z-w|}\right) & \mu(z,\frac{1}{\tau}) \ge |z-w| \\ e^{-C_2\frac{|z-w|}{\mu(z,\frac{1}{\tau})}} & \mu(z,\frac{1}{\tau}) \le |z-w|\end{cases}$$

Proof. We just need to integrate in s for the estimate. Let $\delta > 0$. Then

$$\int_0^\infty H_{\tau p}(s,x,y)\,ds \le \int_0^\delta \frac{1}{s}e^{-c_2\frac{|x-y|^2}{s}}\,ds + \int_\delta^\infty \frac{1}{s}e^{-c_3\frac{s}{\mu(x,\frac{1}{\tau})^2}}\,ds = I + II.$$

To estimate I, we let $t = c_2\frac{|x-y|^2}{s}$, so $-\frac{1}{t}\,dt = \frac{1}{s}\,ds$ and

$$I = \int_{\frac{|x-y|^2}{\delta}}^\infty \frac{1}{t}e^{-t}\,dt.$$

If $c_2\frac{|x-y|^2}{\delta} \le 1$, then

$$I = \int_{c_2\frac{|x-y|^2}{\delta}}^1 \frac{1}{t}e^{-t}\,dt + \int_1^\infty \frac{1}{t}e^{-t}\,dt \le C\left(\log\left(\frac{\delta}{|x-y|^2}\right)+1\right).$$

Also, if $c_2\frac{|x-y|^2}{\delta} \ge 1$,

$$I \le \frac{1}{c_2\frac{|x-y|^2}{\delta}}\int_{c_2\frac{|x-y|^2}{\delta}}^\infty e^{-t}\,dt = C\frac{\delta}{|x-y|^2}e^{-c_2\frac{|x-y|^2}{\delta}} \le Ce^{-c_2\frac{|x-y|^2}{\delta}}.$$

To estimate II, set $t = c_3\frac{s}{\mu(x,\frac{1}{\tau})^2}$, and we have

$$II = \int_{c_3\frac{\delta}{\mu(x,\frac{1}{\tau})^2}}^\infty \frac{1}{t}e^{-t}\,dt.$$

If $c_3\frac{\delta}{\mu(x,\frac{1}{\tau})^2} \le 1$, we have

$$II = \int_{c_3\frac{\delta}{\mu(x,\frac{1}{\tau})^2}}^1 \frac{1}{t}e^{-t}\,dt + \int_1^\infty \frac{1}{t}e^{-t}\,dt \le C\left(\log\left(\frac{\mu(x,\frac{1}{\tau})^2}{\delta}\right)+1\right).$$

Also, if $c_3\frac{\delta}{\mu(x,\frac{1}{\tau})^2} \ge 1$,

$$II \le \left(c_3\frac{\delta}{\mu(x,\frac{1}{\tau})^2}\right)^{-1}\int_{c_3\frac{\delta}{\mu(x,\frac{1}{\tau})^2}}^\infty e^{-t}\,dt = \frac{\mu(x,\frac{1}{\tau})^2}{\delta}e^{-c_3\frac{\delta}{\mu(x,\frac{1}{\tau})^2}} \le Ce^{-c_3\frac{\delta}{\mu(x,\frac{1}{\tau})^2}}.$$

Setting $\delta = \frac{|x-y|}{\mu(x,\frac{1}{\tau})}$ yields the result. $\square$

Appendix A

Kurata's Subsolution Estimate

The goal of this appendix is to prove a version of the subsolution estimate from [Kur00]. We will prove:

Lemma A.1. *If* $(s_0, z_0) \in (0, \infty) \times \mathbb{C}$ *and* $u(s, z)$ *is a* C^2 *solution of*

$$\frac{\partial u}{\partial s} + \Box_{\tau p} u = 0$$

on $Q_{2r}(s_0, z_0)$. *Then if* $\tau > 0$, *there exists* $C > 0$ *so that*

$$\sup_{(s,z) \in Q_{r/2}(s_0,z_0)} |u(s, z)| \leq \frac{C}{r^2} \iint_{Q_{2r/3}(s_0,z_0)} |u(s, z)|^2 \, dzds.$$

The proof has two parts. First, we show that $\frac{\partial |u|}{\partial s} \leq \triangle |u|$, i.e. $|u|$ is a subsolution of the ordinary heat equation

$$\frac{\partial g}{\partial s} - \triangle g = 0. \tag{A.1}$$

Second, we use the "standard subsolution estimate" for (A.1) to conclude the proof. Let the heat ball $E(s, z; r)$ be

$$E(s, z; r) = \left\{ (t, w) \in \mathbb{R} \times \mathbb{C} : t \leq s \text{ and } \frac{1}{4\pi t} e^{-\frac{|w|^2}{4t}} \geq \frac{1}{r^2} \right\}.$$

If $v(s, z)$ is a subsolution of (A.1) on a neighborhood of $E(s, z; r)$, i.e. $\frac{\partial v}{\partial s} - \triangle v \leq 0$, then the standard subsolution estimate for (A.1) is

$$v(s, z) \leq \frac{1}{4r^2} \iint_{E(s,z;r)} v(t, w) \frac{|w|^2}{t^2} \, dwdt. \tag{A.2}$$

A proof of this inequality can be found in [Eva00] (§2.3 Theorem 3, §2.5 Problem 14).

One result we will need is Kato's inequality (Theorem X.33, [RS75]):

Proposition A.2 (Kato's Inequality)**.** *Let* $f \in L^1_{loc}(\mathbb{C})$, $(X_1^2 + X_2^2)f \in L^2_{loc}$, *and* $\operatorname{sgn} f = \frac{\bar{f}}{|f|}$. *Then*

$$\triangle |f| \geq \operatorname{Re}\big[\operatorname{sgn} f\,(X_1^2 + X_2^2)f\big].$$

Proof of Lemma A.1. By computing $\frac{\partial |u|^2}{\partial s}$, we first note that

$$\frac{\partial |u|}{\partial s} = \operatorname{Re}\Big(\frac{\partial u}{\partial s}\operatorname{sgn} u\Big).$$

Next, a short computation shows $X_1^2 + X_2^2 = -4\Box_{\tau p} + 4\tilde{V}$ where $\tilde{V} = \frac{1}{4}\tau\triangle p$. Then by Kato's inequality,

$$\begin{aligned}\triangle |u| &\geq \operatorname{Re}\big[\operatorname{sgn} u\,(X_1^2 + X_2^2)u\big] = \operatorname{Re}\Big[\operatorname{sgn} u\,(-4\Box_{\tau p} + 4\tilde{V})u\Big] \\ &\geq -4\operatorname{Re}\big(\operatorname{sgn} u\,\Box_{\tau p} u\big) = -4\operatorname{Re}\Big(\operatorname{sgn} u\,\frac{\partial u}{\partial s}\Big) = -4\operatorname{Re}\Big(\frac{\partial |u|}{\partial s}\Big).\end{aligned}$$

By (A.2),

$$|u(s,z)| \leq \frac{1}{4r^2}\iint_{E(s,z;r)} |u(t,w)|\frac{|w|^2}{t^2}\,dwdt.$$

To finish the proof, observe that for some constant C, $E(s,z;r) \subset Q_{Cr}(s,z)$. The result is proved by combining this fact with Hölder's inequality and noting that

$$\iint_{E(s,z;r)} \frac{|w-z|^4}{(t-s)^4}\,dwdt = \iint_{E(0,0;r)} \frac{|w|^4}{t^4}\,dwdt = \iint_{E(0,0;1)} \frac{|y|^4}{\tilde{t}^4}\,dyd\tilde{t} = c.$$

under the change of variables $w = ry$ and $t = r^2\tilde{t}$. $\square$

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